Analytic Composition Expansions About Functional Equation Fixed Points

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Abstract

Given holomorphic functions satisfying the functional equation $\phi = \sigma \circ \phi \circ \tau$ where τ has an attracting fixed point paired with a repelling fixed point of σ, we prove ϕ can be expressed as a composition expansion $\lim_{n\to\infty} \sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ where ψ approximates ϕ in some sense. With certain restrictions, ϕ is the unique function satisfying the functional equation. Conversely, given a functional equation of the specified form, we construct a function which satisfies it. The idea behind the proof is to view the transformation $f \to \sigma \circ f \circ \tau$ as a contraction mapping on a particular space of holomorphic functions. As a basic example, the functional equation $\cos z = 2\cos^2\left(\frac{z}{2}\right) - 1$ generates a composition expansion for $\cos z$.

Introduction

Let $f^{\circ k}$ denote k iterated compositions of any function $f : X \to X$ where $f^{\circ 0} := \mathrm{id}_X$ and $f^{\circ(k+1)} = f \circ f^{\circ k}$. Where applicable, let $f^{(k)}$ denote the k^{th} derivative of f. Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$. Unless we explicitly mention the extended complex plane $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ as in Theorem 2.15 and Theorem 3.5, it should be assumed all statements refer to $\mathbb{C}$ where applicable. The entirety of the discussion presented here was originally motivated by proving and generalizing convergence of the following example.

$$\forall x \in \mathbb{R} \qquad \cos x = \lim_{n\to\infty} \psi_n(x) \qquad \psi_0(x) := 1 - \frac{1}{2}x^2 \qquad \psi_{n+1}(x) := 2\psi_n^2\left(\frac{x}{2}\right) - 1$$

We discuss some intuition for why ψ_n should converge to cos. Notice cos is a fixed point of the recurrence relation given for ψ_n. Also notice ψ_0 is a small-angle approximation for cos. The idea is to use the double angle identity to increase the accuracy of the small angle approximation. Suppose we want to compute $\cos x$ for some fixed $x \neq 0$. Since x is away from zero, $\psi_0(x)$ is likely a poor approximation. Using the formula $\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$, observe the value of $\cos x$ is determined by the value of $\cos\left(\frac{x}{2}\right)$. Since $\frac{x}{2}$ is closer to zero, we can expect $\psi_0\left(\frac{x}{2}\right)$ to be a decent approximation of $\cos\left(\frac{x}{2}\right)$. Then use the double angle identity to convert the approximate value of $\cos\left(\frac{x}{2}\right)$ to an approximate value of $\cos x$. In other words, $\psi_1(x)$ should be a better approximation of $\cos x$ than $\psi_0(x)$. We repeat this procedure by continually halving the angle until it is sufficiently close to zero for the small angle approximation to have the desired level of accuracy.

Naturally, this intuition leads to some questions. Can we use other multiple angle identities of cos in a similar way (e.g. $\cos 3x = 4\cos^3 x - 3\cos x$)? Which ψ_0 should the sequence converge for? The intuition relies on ψ_0 approximating cos near $x = 0$ in some sense. To what other situations can this iterated mapping principle be applied? Similar ideas are commonly employed in algorithms meant to compute elementary functions [3].

Before we attempt to answer these questions, we rigorously prove the convergence of ψ_n to cos. We intend to view the transformation L which takes $\psi_n \to \psi_{n+1}$ as a contraction mapping on a certain space of holomorphic functions. We derive a formula for L in terms of power series coefficients. Since power series are absolutely convergent within their radius of convergence, the below manipulations are justified.

$$L\left(\sum_{k=0}^{\infty} a_k z^k\right) = -1 + 2\left(\sum_{k=0}^{\infty} a_k \left(\frac{z}{2}\right)^k\right)^2 = -1 + 2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{a_i a_j z^{i+j}}{2^{i+j}} = -1 + 2\sum_{k=0}^{\infty} \frac{z^k}{2^k} \sum_{\ell=0}^{k} a_\ell a_{k-\ell}$$

In the last step, we are simply collecting similar powers of z (i.e. $k = i + j$). We use the following norm to induce a metric:

$$\left\| \sum_{k=0}^{\infty} a_k z^k \right\| := \sum_{k=0}^{\infty} |a_k| \qquad\qquad d(f,g) := \|f - g\|$$

We intend to prove L is a contraction map on the following space endowed with the previous metric.

$$\mathcal{S} := \{ \ \psi_n \mid n \in \mathbb{N} \ \} \cup \{\cos\}$$

For the metric on $\mathcal{S}$ to be well-defined, we need to ensure $\|f\| < \infty$ for all $f \in \mathcal{S}$. Since each ψ_n is a polynomial, $\|\psi_n\| < \infty$ clearly. We know $\|\cos\| < \infty$ because the power series for cos is absolutely convergent. This ensures the metric exists because $d(f,g) \le \|f\| + \|g\|$. Also note $L : S \to S$ by construction. Before we get to the main proof, we prove a few lemmas. For simplicity, let $\psi_{n,k}$ be the k^{th} coefficient of ψ_n.

$$\forall n \in \mathbb{N} \qquad \psi_n(z) = \sum_{k=0}^{\infty} \psi_{n,k} z^k$$

We claim the odd coefficients are zero for every function in $\mathcal{S}$. The base case ψ_0 and cos is trivial. For the inductive step, assume the statement for ψ_n and prove it for ψ_{n+1}. Look at the formula we derived for L. For $k \ge 1$, we have $\psi_{n+1,k} = 2^{1-k} \sum_{\ell=0}^{k} \psi_{n,\ell}\psi_{n,k-\ell}$. Whenever k is odd, either ℓ or $k - \ell$ is odd. This observation immediately proves the claim. Similarly, it is not hard to prove $\psi_{n,0} = 1$ and $\psi_{n,2} = -\frac{1}{2}$ for all $n \in \mathbb{N}$. As a consequence, many terms of the distance formula on $\mathcal{S}$ are zero.

$$\forall f, g \in \mathcal{S} \qquad d(f,g) = \sum_{k=0}^{\infty} |f_k - g_k| = \sum_{k=2}^{\infty} |f_{2k} - g_{2k}|$$

The formula for $L : S \to S$ also simplifies.

$$L(f) = 1 - \frac{1}{2}z^2 + 2\sum_{k=2}^{\infty} \frac{z^{2k}}{4^k} \sum_{\ell=0}^{k} f_{2\ell} f_{2k-2\ell}$$

We claim every power series coefficient of an element of $\mathcal{S}$ has absolute value less than or equal to 1. This is trivial for ψ_0 and cos. We use induction. Assuming $|\psi_{n,k}| \le 1$ for all $k \in \mathbb{N}$, we prove $|\psi_{n+1,k}| \le 1$ for all $k \in \mathbb{N}$. Due to previous lemmas about $\psi_{n,k}$, we only have to check $|\psi_{n+1,2k}| \le 1$ for $k \ge 2$.

$$\forall k \ge 2 \qquad |\psi_{n+1,2k}| = \left| 2^{1-2k} \sum_{\ell=0}^{k} \psi_{n,2\ell}\psi_{n,2k-2\ell} \right| \le 2^{1-2k} \sum_{\ell=0}^{k} |\psi_{n,2\ell}|\,|\psi_{n,2k-2\ell}| \le 2^{1-2k} \sum_{\ell=0}^{k} 1 = 2^{1-2k}(k+1) \le 1$$

We are ready to prove L is a contraction mapping on $\mathcal{S}$.

$$\forall f, g \in \mathcal{S} \qquad d(Lf, Lg) = \sum_{k=2}^{\infty} |(Lf)_{2k} - (Lg)_{2k}| = \sum_{k=2}^{\infty} \frac{2}{4^k} \left| \sum_{\ell=0}^{k} (f_{2\ell} f_{2k-2\ell} - g_{2\ell} g_{2k-2\ell}) \right| =$$

$$\sum_{k=2}^{\infty} \frac{2}{4^k} \left| \sum_{\ell=0}^{k} (f_{2\ell} f_{2k-2\ell} - f_{2\ell} g_{2k-2\ell} + f_{2\ell} g_{2k-2\ell} - g_{2\ell} g_{2k-2\ell}) \right| \le \sum_{k=2}^{\infty} \frac{2}{4^k} \sum_{\ell=0}^{k} \Big(|f_{2\ell}|\,|f_{2k-2\ell} - g_{2k-2\ell}| + |g_{2k-2\ell}|\,|f_{2\ell} - g_{2\ell}| \Big) \le$$

$$\sum_{k=2}^{\infty} \frac{2}{4^k} \sum_{\ell=0}^{k} \Big(|f_{2k-2\ell} - g_{2k-2\ell}| + |f_{2\ell} - g_{2\ell}| \Big) = \sum_{k=2}^{\infty} \frac{4}{4^k} \sum_{\ell=0}^{k} |f_{2\ell} - g_{2\ell}| \le \sum_{k=2}^{\infty} \frac{4}{4^k} \sum_{\ell=0}^{\infty} |f_\ell - g_\ell| = d(f,g) \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3} d(f,g)$$

We have proven $d(Lf, Lg) \le \frac{1}{3} d(f,g)$ for all $f, g \in \mathcal{S}$. Notice $d(\psi_n, \cos) = d(L^{\circ n}\psi_0, L^{\circ n}\cos) \le \frac{1}{3^n} d(\psi_0, \cos)$. Therefore $\lim_{n\to\infty} d(\psi_n, \cos) = 0$, which is the definition of convergence in $\mathcal{S}$. We are not done yet. The goal was to prove pointwise convergence of ψ_n to cos. Let c_k be the k^{th} coefficient of the cosine power series. Take any $z \in \mathbb{C}$ such that $|z| \le 1$.

$$\forall n \in \mathbb{N} \qquad 0 \le |\psi_n(z) - \cos z| = \left|\sum_{k=0}^{\infty}(\psi_{n,k} - c_k)z^k\right| \le \sum_{k=0}^{\infty} |\psi_{n,k} - c_k|\,|z|^k \le \sum_{k=0}^{\infty} |\psi_{n,k} - c_k| = d(\psi_n, \cos)$$

Take the limit $n \to \infty$ and use the squeeze theorem to conclude $\lim_{n\to\infty} |\psi_n(z) - \cos z| = 0$ uniformly on the ball $|z| \le 1$. Thus ψ_n uniformly converges to cos on $\{ z \in \mathbb{C} \mid |z| \le 1 \}$ as $n \to \infty$. To get uniform-on-compacts convergence on the entire complex plane, we perform a trick. For simplicity, define $\sigma(z) := 2z^2 - 1$ so that $\psi_{n+1}(z) = \sigma \circ \psi_n\left(\frac{z}{2}\right)$. Take any $z \in \mathbb{C}$. Choose $k \in \mathbb{N}$ large enough to force $\left|\frac{z}{2^k}\right| \le 1$. We have already proven the following.

$$\lim_{n\to\infty} \psi_n\left(\frac{z}{2^k}\right) = \cos\left(\frac{z}{2^k}\right)$$

Apply $\sigma^{\circ k}$ to both sides, use continuity of σ to move the limit, and simplify.

$$\sigma^{\circ k}\left(\lim_{n\to\infty} \psi_n\left(\frac{z}{2^k}\right)\right) = \sigma^{\circ k} \circ \cos\left(\frac{z}{2^k}\right) \quad \Longrightarrow \quad \lim_{n\to\infty} \sigma^{\circ k} \circ \psi_n\left(\frac{z}{2^k}\right) = \cos z \quad \Longrightarrow$$

$$\lim_{n\to\infty} \psi_{n+k}(z) = \cos z \quad \Longrightarrow \quad \lim_{n\to\infty} \psi_n(z) = \cos z$$

Result (1)**.** $\psi_n : \mathbb{C} \to \mathbb{C}$ *as defined below has uniform-on-compacts convergence to cosine.*

$$\forall z \in \mathbb{C} \qquad \psi_0(z) := 1 - \frac{1}{2}z^2 \qquad \psi_{n+1}(z) := 2\psi_n^2\left(\frac{z}{2}\right) - 1$$

More generally, we are interested in the convergence of $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ to another function ϕ as $n \to \infty$ where τ has an attracting fixed point z_0 and σ has a repelling fixed point $w_0 := \psi(z_0)$. In analogy to the previous example, ψ is meant to approximate ϕ near z_0, and ϕ should satisfy $\phi = \sigma \circ \phi \circ \tau$. We want τ to have an attracting fixed point at z_0 so that we may iterate $\tau^{\circ n}(z)$ to increase the accuracy of the approximation ψ. It is not completely obvious why σ should have a repelling fixed point at w_0, but the idea is that the repulsiveness of w_0 should perfectly cancel the attractiveness of z_0. If z_0 is too attractive, we might expect $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ to converge to the trivial solution $\phi(z) = w_0$ everywhere. On the other hand, if w_0 is too repulsive, we might expect the sequence of functions to diverge.

We mention some known results which are related to our ideas. Koenigs linearization theorem gives convergence of $\lambda^{-n}\tau^{\circ n}(z)$ when $\tau(0) = 0$, $\tau'(0) = \lambda$, and $|\lambda| < 1$ [7]. In our presentation, this is the convergence of $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ with $\psi(z) = z$ and $\sigma(z) = \lambda^{-1}z$. Koenigs theorem will be useful for replacing $\tau(z)$ with its linearization λz as in Lemma 2.2.

The Schröder/Poincaré functional equation $\phi(\lambda z) = \sigma(\phi(z))$ with $\sigma(0) = 0$, $\sigma'(0) = \lambda$, and $|\lambda| > 1$ is also relevant [1]. Converting to our formulation, we have $\phi = \sigma \circ \phi \circ \tau$ with $\tau(z) := \lambda^{-1}z$. When σ is a rational function, there is a meromorphic solution ϕ, often referred to as a Poincaré function [10, 15, 1, 11, 5, 6]. When σ is a polynomial, $\sigma^{\circ n}(\lambda^{-n}z)$ converges uniformly on compacts to an entire Poincaré function (i.e. $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ converges with $\psi(z) = z$) [8]. It is unclear whether or not it is known that all Poincaré functions admit such a composition expansion, but this will be a consequence of our theorems (c.f. Theorem 2.15 and Theorem 3.4).

Prior literature is primarily concerned with whether a non-trivial solution ϕ of $\phi = \sigma \circ \phi \circ \tau$ exists for given σ, τ. We've chosen to view the problem from a different perspective where emphasis is instead placed on convergence of the composition expansion $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ for suitably chosen ψ. Our approach may be useful for those who need to practically compute approximations to ϕ (c.f. Corollary 3.1 and Theorem 3.4).

Our approach can also be used to approximate ϕ when the roles of τ and σ are switched. If τ has a repelling fixed point and σ has an attracting fixed point, then the functional equation $\phi = \sigma \circ \phi \circ \tau$ should be converted to $\sigma^{-1} \circ \phi \circ \tau^{-1} = \phi$ using local inverses of τ and σ. In which case, ϕ may be approximated as the composition expansion $\sigma^{\circ(-n)} \circ \psi \circ \tau^{\circ(-n)}$ for suitably chosen ψ.

Proof Generalization

This section is devoted to generalizing the proof given in the introduction. Recall the following details from complex analysis [14, Chapter 10]. A function $f : U \to \mathbb{C}$ is holomorphic on an open set $U \subset \mathbb{C}$ when f is differentiable on U. f is entire when f is holomorphic on $U = \mathbb{C}$. f is holomorphic at a point $z_0 \in \mathbb{C}$ when f is holomorphic on some neighborhood of z_0. If f is holomorphic on a ball $B_r(z_0)$, then f has a power series expansion about z_0 with radius of convergence at least r. Power series are absolutely convergent and may be differentiated term-wise within their radii of convergence. If f is holomorphic at z_0 and $f'(z_0) \neq 0$, then f has a local inverse which is holomorphic at z_0. If a sequence of holomorphic functions $\{f_n : U \to \mathbb{C}\}_{n=0}^{\infty}$ converges uniformly on every compact set $K \subset U$, then the pointwise limit $f_\infty(z) := \lim_{n\to\infty} f_n(z)$ is holomorphic on U, and the sequence of derivatives $\{f'_n\}_{n=0}^{\infty}$ converges to f'_∞ uniformly on compacts $K \subset U$. Given f and g which are holomorphic on an open and connected set U, $f = g$ on all of U whenever $f = g$ on some subset of U which has an accumulation point (identity prinicple). The proof of the following theorem will take up the majority of this section.

Theorem (2)**.** *Suppose ψ and τ are holomorphic at $z_0 \in \mathbb{C}$. Suppose σ is holomorphic at $w_0 := \psi(z_0)$. Let $\lambda := \tau'(z_0)$. Suppose $\tau(z_0) = z_0$, $\sigma(w_0) = w_0$, and $0 < |\lambda| < 1$. Define $\psi_n = \sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ for all $n \in \mathbb{N}$. Suppose there exists $m \in \mathbb{Z}^+$ such that $|\lambda|^m |\sigma'(w_0)| < 1$ and $\psi_0^{(k)}(z_0) = \psi_1^{(k)}(z_0)$ for all $k < m$.*

Then there exists an open disc $\mathcal{D}$ of positive radius centered at z_0 such that $\tau(\mathcal{D}) \subset \mathcal{D}$ where $\{\psi_n\}_{n=0}^{\infty}$ converges uniformly on $\mathcal{D}$ to a holomorphic function ϕ. Furthermore, ϕ is the unique holomorphic function on $\mathcal{D}$ satisfying $\phi = \sigma \circ \phi \circ \tau$ and $\phi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$ (including $k = 0$).

We characterize the conditions under which ϕ is trivial. If $\lambda^k \sigma'(w_0) \neq 1$ for all $k \in \mathbb{Z}^+$, then $\phi(z) = w_0$ for all $z \in \mathcal{D}$ (i.e. the trivial solution). If $\lambda^\ell \sigma'(w_0) = 1$ for some $\ell \in \mathbb{Z}^+$, then ϕ is trivial if and only if $\psi^{(k)}(z_0) = 0$ for all k with $0 < k \leq \ell$. [5, Lemma 2.1] was helpful in determining this characterization of trivial solutions.

(General Remarks) z_0 is an attracting fixed point of τ, and w_0 is a repelling fixed point of σ (assuming ϕ is non-trivial). The definitions of ψ and τ do not matter outside of small neighborhoods of z_0. Similarly, σ does not matter outside of a neighborhood of w_0. The assumption $\psi_0^{(k)}(z_0) = \psi_1^{(k)}(z_0)$ for all $k < m$ is actually a way of ensuring $\psi_0^{(k)}(z_0) = \psi_n^{(k)}(z_0)$ for all $k < m$ and $n \in \mathbb{N}$. The assumption $|\lambda|^m |\sigma'(w_0)| < 1$ is a sort of regularity condition to ensure convergence. The region of convergence $\mathcal{D}$ may be extended given additional assumptions. See Theorem 2.14 and Theorem 2.15 for details. For specific examples, refer to Example 3.2, Example 3.3, Example 3.6, and Example 3.7.

Assume $m \geq 2$ for the duration of this paragraph. We claim the derivatives $\tau^{(k)}(z_0)$, $\psi^{(k)}(z_0)$, and $\sigma^{(k)}(w_0)$ for $k \geq m$ may be modified (i.e. alter the power series expansions), and the hypotheses of Theorem 2 will continue to be satsified. We verify $\psi_0^{(k)}(z_0) = \psi_1^{(k)}(z_0)$ continues to be satisfied for $k < m$ after modifying the higher derivatives. Consider fully expanding the k^{th} derivative (for some $k < m$) of $\sigma \circ \psi \circ \tau$ evaluated at z_0 using the chain rule and product rule. The resulting expression will not include any derivatives of σ, ψ, or τ with order $\geq m$. The claim follows. So any tuple $(\sigma, \psi, \tau, z_0, m)$ satisfying Theorem 2 immediately yields infinitely many other tuples satisfying Theorem 2. Consider grouping all such tuples into an equivalence class. The prototypical tuple from a given equivalence class may be taken where σ, ψ, τ are polynomials of degree at most $m - 1$ (set all higher order terms to zero).

Before diving into the weeds, we prove part of the trivial ϕ characterization (assuming everything else has been proven). Suppose $\lambda^\ell \sigma'(w_0) = 1$ for some $\ell \in \mathbb{Z}^+$. Then the theorem applies for $m := \ell + 1$. If $\psi^{(k)}(z_0) = 0$ for all k with $0 < k \leq \ell$, then $\phi(z) = w_0$ for all $z \in \mathcal{D}$ by uniqueness (the constant function w_0 satisfies the conditions which uniquely define ϕ). If $\psi^{(j)}(z_0) \neq 0$ for some j with $0 < j \leq \ell$, then $\phi^{(j)}(z_0) = \psi^{(j)}(z_0) \neq 0$, so ϕ is non-trivial. Hence we've shown ϕ is trivial if and only if $\psi^{(k)}(z_0) = 0$ for all k with $0 < k \leq \ell$. Now we begin our crawl through the weeds.

ϕ is defined to be the pointwise limit $\phi(z) := \lim_{n\to\infty} \psi_n(z)$ when the limit exists. The theorem guarantees the limit exists and occurs uniformly on some disc $\mathcal{D}$ centered at z_0. We specify $\tau(\mathcal{D}) \subset \mathcal{D}$ so that it makes sense to have $\phi = \sigma \circ \phi \circ \tau$ when ϕ is only necessarily defined on $\mathcal{D}$. Instead of directly attacking the proof, we first perform some preliminary simplifications. Koenigs theorem will be used to replace τ with its linearization. Feel free to skip lemmas and come back to them as they are used.

Theorem (2.1)**.** *[Koenigs Linearization] Suppose τ is holomorphic at the origin and satisfies $\tau(0) = 0$ where $\lambda := \tau'(0)$ and $0 < |\lambda| < 1$. Define $\tau_n(z) = \lambda^{-n}\tau^{\circ n}(z)$ for all $n \in \mathbb{N}$. Then there exists an open disc D of positive radius R centered at the origin such that $|\tau(z)| \leq |z|$ for all $z \in D$ where $\{\tau_n\}_{n=0}^{\infty}$ converges uniformly on D to a holomorphic function φ. Furthermore $\varphi(0) = 0$, $\varphi'(0) = 1$, and $\varphi(\tau(z)) = \lambda\varphi(z)$ for all $z \in D$.*

Define $C := \frac{\tau''(0)}{2!}$. Considering the power series expansion of τ about the origin, we have the following limits:

$$\lim_{z\to 0} \frac{\tau(z)}{z} = \lambda \qquad \lim_{z\to 0} \frac{\tau(z) - \lambda z}{z^2} = C$$

Defining $\epsilon_1 := \frac{1-|\lambda|}{2}$, the first limit implies we can find $\delta_1 > 0$ such that $|\tau(z)| \leq (|\lambda| + \epsilon_1)\,|z| \leq |z|$ for all z with $|z| < \delta_1$. Defining $\epsilon_2 := 1$, the second limit implies we can find $\delta_2 > 0$ such that $|\tau(z) - \lambda z| \leq (|C| + \epsilon_2)\,|z|^2$ for all z with $|z| < \delta_2$. By assumption, there exists $\delta_3 > 0$ such that τ is holomorphic on the open disc of radius δ_3 centered at the origin. We claim $\{\tau_n\}_{n=0}^{\infty}$ converges uniformly on an open disc D of radius R (defined below) centered at the origin.

$$R := \min\left(\delta_1,\ \delta_2,\ \delta_3,\ \frac{|\lambda|\,\big(1-|\lambda|\,\big)}{4\big(1+|C|\,\big)}\right)$$

Since $R \leq \delta_1$, we have $|\tau(z)| \leq |z|$ for all $z \in D$, which implies $\tau(D) \subset D$. As a consequence, we must have $|\tau^{\circ n}(z)| < \delta_2$ for all $n \in \mathbb{N}$ and $z \in D$. Using the inequality derived from the second limit, we have $\left|\tau^{\circ(n+1)}(z) - \lambda\tau^{\circ n}(z)\right| \leq (|C| + \epsilon_2)\,|\tau^{\circ n}(z)|^2$ for all $n \in \mathbb{N}$ and $z \in D$. Divide both sides by λ^{n+1} and rewrite in terms of τ_n and τ_{n+1}.

$$\forall n \in \mathbb{N} \quad \forall z \in D \qquad |\tau_{n+1}(z) - \tau_n(z)| \leq \big(1 + |C|\,\big)\,|\lambda|^{n-1}\,|\tau_n(z)|^2$$

Lemma (2.1.1)**.** *Induction may be used to establish the following inequality.*

$$\forall n \in \mathbb{N} \quad \forall z \in D \qquad |\tau_n(z)| \leq 2\,|z|$$

The base case $|z| \leq 2\,|z|$ with $n = 0$ is trivial. Assuming the statement holds for all $n < m$, we prove it for m.

$$\forall z \in D \qquad |\tau_m(z)| = \left|z + \sum_{n=0}^{m-1}\big(\tau_{n+1}(z) - \tau_n(z)\big)\right| \leq |z| + \sum_{n=0}^{m-1}|\tau_{n+1}(z) - \tau_n(z)| \leq |z| + \big(1+|C|\,\big)\sum_{n=0}^{m-1}|\lambda|^{n-1}\,|\tau_n(z)|^2 \leq$$

$$|z| + 4\,|z|^2\big(1+|C|\,\big)\sum_{n=0}^{m-1}|\lambda|^{n-1} \leq |z| + |z|^2\,\frac{4\big(1+|C|\,\big)}{|\lambda|}\sum_{n=0}^{\infty}|\lambda|^n = |z| + |z|^2\,\frac{4\big(1+|C|\,\big)}{|\lambda|\,\big(1-|\lambda|\,\big)} \leq |z| + |z| = 2\,|z|$$

In the last step, we are using the bound $z \leq \frac{|\lambda|(1-|\lambda|)}{4(1+|C|)}$ given by the definition of R. Thus the lemma is proven. As an immediate corollary, we have $|\tau_n(z)| \leq 2R$ for all $n \in \mathbb{N}$ and $z \in D$. Combining this corollary with a previous inequality, we get:

$$\forall n \in \mathbb{N} \quad \forall z \in D \qquad |\tau_{n+1}(z) - \tau_n(z)| \leq 4R^2\big(1+|C|\,\big)\,|\lambda|^{n-1}$$

Since $\sum_{n=0}^{\infty} 4R^2\big(1+|C|\,\big)\,|\lambda|^{n-1} < \infty$, the Weierstrass M-test implies $\tau_m(z) = z + \sum_{n=0}^{m-1}\big(\tau_{n+1}(z) - \tau_n(z)\big)$ converges uniformly on D as $m \to \infty$. So we may define $\varphi(z) := \lim_{n\to\infty}\tau_n(z)$ for all $z \in D$. Since $R < \delta_3$ and $\tau(D) \subset D$, τ_n is holomorphic on D for each $n \in \mathbb{N}$ by the chain rule. Since a uniform limit of holomorphic functions is holomorphic, φ is holomorphic on D. Clearly $\varphi(0) = 0$. Since $\tau_n'(0) = 1$ for all $n \in \mathbb{N}$, we must have $\varphi'(0) = 1$. We show φ satisfies $\varphi(\tau(z)) = \lambda\varphi(z)$ for all $z \in D$.

$$\forall z \in D \qquad \varphi(\tau(z)) = \lim_{n\to\infty}\lambda^{-n}\tau^{\circ(n+1)}(z) = \lambda\left(\lim_{n\to\infty}\lambda^{-(n+1)}\tau^{\circ(n+1)}(z)\right) = \lambda\varphi(z)$$

Hence we have proven Koenigs theorem. See [4, Theorem 3.2.2] and [7, Theorem 1.2] for alternate proofs. φ is actually the unique holomorphic function on D satisfying $\varphi'(0) = 1$ and $\varphi(\tau(z)) = \lambda\varphi(z)$ for all $z \in D$. However, it is unnecessary to prove uniqueness here. Once Theorem 2 is proven, uniqueness of φ will follow, since Koenigs theorem is the special case $\psi(z) = z$ and $\sigma(z) = \lambda^{-1}z$ with $z_0 = w_0 = 0$ and $m = 2$.

The result $|\tau(z)| \leq |z|$ for all $z \in D$ is important for the next lemma. In particular, it is implied that $\tau(D') \subset D'$ for all discs D' centered at the origin with radii smaller than R. This allows us to shrink D to any desired sufficiently small size while retaining τ invariance. The next lemma shifts the fixed points z_0 and w_0 to zero and uses Koenigs theorem to replace τ with its linearization.

Lemma (2.2)**.** *Without loss of generality in proving Theorem 2, we may restrict to the case $\tau(z) = \lambda z$ with $z_0 = w_0 = 0$.*

Assuming Theorem 2 has been proven in the restricted case, we prove the general formulation. We restate the assumptions of the general formulation: Suppose ψ and τ are holomorphic at $z_0 \in \mathbb{C}$. Suppose σ is holomorphic at $w_0 := \psi(z_0)$. Let $\lambda := \tau'(z_0)$. Suppose $0 < |\lambda| < 1$, $\tau(z_0) = z_0$, and $\sigma(w_0) = w_0$. Define $\psi_n = \sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ for all $n \in \mathbb{N}$. Suppose there exists $m \in \mathbb{Z}^+$ such that $|\lambda|^m |\sigma'(w_0)| < 1$ and $\psi_0^{(k)}(z_0) = \psi_1^{(k)}(z_0)$ for all $k < m$.

Define $T(z) := \tau(z + z_0) - z_0$ so that $T(0) = 0$ and $T'(0) = \lambda$. Since τ is holomorphic at z_0, T is holomorphic at the origin. Koenigs theorem gives us a function φ holomorphic on an open disc D centered at the origin such that $\varphi(0) = 0$, $\varphi'(0) = 1$, and $\varphi(T(z)) = \lambda\varphi(z)$ for all $z \in D$. Since $\varphi'(0) \neq 0$, there exists a local inverse φ^{-1} holomorphic on the image $\varphi(D')$ of a (smaller) disk $D' \subset D$ centered at the origin. Make the following definitions:

$$S(z) := \sigma(z + w_0) - w_0 \qquad P(z) := \psi\big(\varphi^{-1}(z) + z_0\big) - w_0 \qquad P_n(z) := S^{\circ n} \circ P(\lambda^n z)$$

By construction, we have $P_n(\varphi(z)) = \psi_n(z + z_0) - w_0$ for all $n \in \mathbb{N}$ and $z \in D'$. This is one place where we use the invariance $T(D') \subset D'$ (since φ^{-1} is only defined on $\varphi(D')$). As an example, consider the following computation:

$$P_1(\varphi(z)) = S \circ P(\lambda\varphi(z)) = S \circ P \circ \varphi(T(z)) = S\big(\psi(T(z) + z_0) - w_0\big) = \sigma \circ \psi \circ \tau(z + z_0) - w_0 = \psi_1(z + z_0) - w_0$$

We also have $P(0) = S(0) = 0$ and $S'(0) = \sigma'(w_0)$. Clearly S and P are holomorphic at the origin. The final condition to check is $P_0^{(k)}(0) = P_1^{(k)}(0)$ for all $k < m$. The $k = 0$ case is trivial. Rewriting P_0 and P_1, we need to show:

$$\forall k \in \{1\ ,\ \ldots\ ,\ m-1\} \qquad \frac{d^k}{dz^k}\psi_0\big(\varphi^{-1}(z) + z_0\big)\bigg|_{z=0} = \frac{d^k}{dz^k}\psi_1\big(\varphi^{-1}(z) + z_0\big)\bigg|_{z=0}$$

If $P_0^{(k)}(0)$ is fully expanded using the chain rule and product rule, it will rely on the first k derivatives of ψ_0 and φ^{-1} evaluated at z_0 and 0 respectively. Similarly, $P_1^{(k)}(0)$ relies on the first k derivatives of ψ_1 and φ^{-1} evaluated at z_0 and 0 respectively. With the assumption that $\psi_0^{(k)}(z_0) = \psi_1^{(k)}(z_0)$ for all $k < m$, it should be clear that $P_0^{(k)}(0) = P_1^{(k)}(0)$ for all $k < m$.

Hence we may invoke the restricted version of the theorem. There exists an open disc $\mathcal{D}$ centered at the origin such that $\{P_n\}_{n=0}^{\infty}$ converges uniformly on $\mathcal{D}$ to a holomorphic function ϕ. Furthermore, ϕ is the unique holomorphic function on $\mathcal{D}$ satisfying $\phi(z) = S \circ \phi(\lambda z)$ for all $z \in \mathcal{D}$ and $\phi^{(k)}(0) = P^{(k)}(0)$ for all $k < m$ (including $k = 0$).

Since $\psi_n(z) = P_n \circ \varphi(z - z_0) + w_0$, it must be that $\{\psi_n\}_{n=0}^{\infty}$ converges uniformly to $\phi_2(z) := \phi \circ \varphi(z - z_0) + w_0$ on some disc centered at z_0. The chain rule implies ϕ_2 is holomorphic at z_0. We verify ϕ_2 satisfies $\phi_2 = \sigma \circ \phi_2 \circ \tau$ in some sufficiently small neighborhood of z_0.

$$\phi_2(z) = \phi \circ \varphi(z - z_0) + w_0 = S \circ \phi\big(\lambda\varphi(z - z_0)\big) + w_0 = S \circ \phi \circ \varphi \circ T(z - z_0) + w_0 = \sigma\big(\phi \circ \varphi(\tau(z) - z_0) + w_0\big) = \sigma \circ \phi_2 \circ \tau(z)$$

Next, we show $\phi_2^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$. For the $k = 0$ case, note $\phi_2(z_0) = w_0 = \psi(z_0)$. For $0 < k < m$, we use another chain/product rule symmetry argument. We want to show:

$$\forall k \in \{1\ ,\ \ldots\ ,\ m-1\} \qquad \psi^{(k)}(z_0) = \frac{d^k}{dz^k}P(\varphi(z))\bigg|_{z=0} = \frac{d^k}{dz^k}\phi(\varphi(z))\bigg|_{z=0} = \phi_2^{(k)}(z_0)$$

The first equality holds because $P(\varphi(z)) = \psi(z + z_0) - w_0$ by definition. The third equality also holds by definition. The middle equality holds by the assumption $\phi^{(k)}(0) = P^{(k)}(0)$ for all $k < m$ (using chain/product rule symmetry). Now we show uniqueness of ϕ_2.

Suppose f is holomorphic at z_0 where $f = \sigma \circ f \circ \tau$ and $f^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$. Define $g(z) := f(\varphi^{-1}(z) + z_0) - w_0$. Notice g satisfies $g(z) = S \circ g(\lambda z)$ and $g^{(k)}(0) = P^{(k)}(0)$ for all $k < m$. Since ϕ is the unique function satisfying these conditions, $g = \phi$. Inverting the definition of g, we have $f(z) = g \circ \varphi(z - z_0) + w_0 = \phi \circ \varphi(z - z_0) + w_0 = \phi_2(z)$, which shows uniqueness.

Note that if $\phi(z) = 0$ for all sufficiently small z, then $\phi_2(z) = w_0$ for all z sufficiently close to z_0. Thus our characterization of trivial ϕ extends from the restricted case to the general theorem. Now we prove the restricted case.

Theorem (2.3)**.** *[Restricted Formulation] Suppose σ and ψ are holomorphic at the origin. Suppose $\sigma(0) = \psi(0) = 0$. Suppose $\lambda \in \mathbb{C}$ where $0 < |\lambda| < 1$. Define $\psi_n(z) = \sigma^{\circ n} \circ \psi(\lambda^n z)$ for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$. Suppose there exists $m \in \mathbb{Z}^+$ such that $|\lambda|^m |\sigma'(0)| < 1$ and $\psi_0^{(k)}(0) = \psi_1^{(k)}(0)$ for all $k < m$.*

Then there exists an open disc $\mathcal{D}$ of positive radius r centered at the origin such that $\{\psi_n\}_{n=0}^{\infty}$ converges uniformly on $\mathcal{D}$ to a holomorphic function ϕ. Furthermore, ϕ is the unique holomorphic function on $\mathcal{D}$ satisfying $\phi(z) = \sigma \circ \phi(\lambda z)$ for all $z \in \mathcal{D}$ and $\phi^{(k)}(0) = \psi^{(k)}(0)$ for all $k < m$ (including $k = 0$). If $\lambda^k \sigma'(0) \neq 1$ for all $k \in \mathbb{Z}^+$, then $\phi(z) = 0$ for all $z \in \mathcal{D}$ (i.e. the trivial solution).

Let σ_i be the i^{th} coefficient of the power series expansion of σ about the origin. By chain rule, ψ_n is holomorphic at the origin for every $n \in \mathbb{N}$. Let $\psi_{n,k}$ be the k^{th} coefficient of the power series expansion of ψ_n about the origin. For any function $f : \mathbb{C} \to \mathbb{C}$, define the operation $(Lf)(z) := \sigma \circ f(\lambda z)$ for all $z \in \mathbb{C}$. Hence $\psi_{n+1} = L\psi_n$ for all $n \in \mathbb{N}$. We will need many lemmas.

Lemma (2.4)**.** *Suppose $n, k \in \mathbb{Z}^+$. The number of k-tuples of positive integers which sum to n is $\binom{n-1}{k-1}$ [2, Corollary 2.5]. The number of k-tuples of non-negative integers which sum to n is $\binom{n+k-1}{k-1}$ [2, Theorem 2.2].*

Lemma (2.5)**.** *For all $z \in \mathbb{C}$ with $|z| < 1$, we have the following:*

$$\left(\frac{1}{1-z}\right)^k = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} z^n$$

$$\left(\frac{1}{1-z}\right)^k = \left(\sum_{j=0}^{\infty} z^j\right)^k = \left(\sum_{j_1=0}^{\infty} z^{j_1}\right) \cdots \left(\sum_{j_k=0}^{\infty} z^{j_k}\right) = \sum_{j_1,\ldots,j_k=0}^{\infty} z^{j_1+\cdots+j_k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} z^n$$

Absolute convergence justifies collecting similar powers ($n = j_1 + \cdots + j_k$). The binomial coefficient comes from the number of ways the k-tuple $(j_1, \ldots, j_k)$ of non-negative integers sums to n.

Lemma (2.6)**.** *Given a function f holomorphic at the origin satisfying $f(0) = 0$ with power series coefficients $\{f_j\}_{j=0}^{\infty}$, the function Lf has power series coefficients $\{(Lf)_k\}_{k=0}^{\infty}$ as described by the below formula.*

$$\forall k \in \mathbb{N} \qquad (Lf)_k = \lambda^k \sum_{i=1}^{k} \sigma_i \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} \prod_{\ell=1}^{i} f_{j_\ell}$$

The middle summation is over all i-tuples $(j_1, \ldots, j_i)$ of positive integers which sum to k. When $k = 0$, the empty summation evaluates to zero by definition. This lemma is related to Faà di Bruno's formula [9].

Since f is holomorphic at the origin, Lf is holomorphic at the origin. Clearly $(Lf)_0 = 0$. Suppose $z \in \mathbb{C}$ is sufficiently small to ensure convergence.

$$\sum_{k=1}^{\infty} (Lf)_k z^k = (Lf)(z) = \sigma \circ f(\lambda z) = \sum_{i=1}^{\infty} \sigma_i \left(\sum_{j=1}^{\infty} f_j \lambda^j z^j\right)^i = \sum_{i=1}^{\infty} \sigma_i \prod_{\ell=1}^{i} \sum_{j_\ell=1}^{\infty} f_{j_\ell} \lambda^{j_\ell} z^{j_\ell} = \sum_{i=1}^{\infty} \sigma_i \sum_{j_1,\ldots,j_i=1}^{\infty} \prod_{\ell=1}^{i} f_{j_\ell} \lambda^{j_\ell} z^{j_\ell} =$$

$$\sum_{i=1}^{\infty}\sigma_i\sum_{k=i}^{\infty}\sum_{\substack{j_1+\cdots+j_i=k\\1\le j\le k}}\prod_{\ell=1}^{i}f_{j_\ell}\lambda^{j_\ell}z^{j_\ell}=\sum_{k=1}^{\infty}\sum_{i=1}^{k}\sigma_i\sum_{\substack{j_1+\cdots+j_i=k\\1\le j\le k}}\prod_{\ell=1}^{i}f_{j_\ell}\lambda^{j_\ell}z^{j_\ell}=\sum_{k=1}^{\infty}\lambda^k z^k\sum_{i=1}^{k}\sigma_i\sum_{\substack{j_1+\cdots+j_i=k\\1\le j\le k}}\prod_{\ell=1}^{i}f_{j_\ell}$$

We have reached the desired expression. To justify collecting powers of z and interchanging the i, k summations, we prove absolute convergence of the multi-sum.

$$\sum_{i=1}^{\infty}\sum_{j_1,\ldots,j_i=1}^{\infty}\left|\sigma_i\prod_{\ell=1}^{i}f_{j_\ell}\lambda^{j_\ell}z^{j_\ell}\right|=\sum_{i=1}^{\infty}|\sigma_i|\left(\sum_{j=1}^{\infty}|f_j|\,|\lambda|^j\,|z|^j\right)^i<\infty$$

The summation over j converges because f is holomorphic at the origin and z is sufficiently small. The j-summation may be considered a power series in its own right, and is therefore continuous in a neighborhood of the origin. The summation over i converges because σ is holomorphic at the origin and z is sufficiently small. Hence the lemma is proven.

Lemma (2.7)**.** *This lemma asserts the uniqueness of ϕ in Theorem 2.3.*

Suppose ϕ is holomorphic at the origin and satisfies $L\phi = \phi$ and $\phi^{(k)}(0) = \psi^{(k)}(0)$ for all $k < m$. Let $\{\phi_k\}_{k=0}^{\infty}$ be the coefficients of the power series expansion of ϕ about the origin. Uniquely specifying ϕ near the origin is equivalent to uniquely specifying all the coefficients ϕ_k. By assumption, $\phi_k = \psi_{0,k}$ for all $k < m$. So we only have to worry about specifying ϕ_k for $k \geq m$. The assumption $L\phi = \phi$ asserts the following by Lemma 2.6.

$$\forall k\in\mathbb{N}\qquad \phi_k=\lambda^k\sum_{i=1}^{k}\sigma_i\sum_{\substack{j_1+\cdots+j_i=k\\1\le j\le k}}\prod_{\ell=1}^{i}\phi_{j_\ell}$$

Currently, ϕ_k is written as a function of $\{\phi_j\}_{j=1}^{k}$. We want to move all the ϕ_k's to one side of the equation. ϕ_k only appears on the right side when $i = 1$.

$$\forall k\in\mathbb{Z}^+\qquad \phi_k=\lambda^k\sigma_1\phi_k+\lambda^k\sum_{i=2}^{k}\sigma_i\sum_{\substack{j_1+\cdots+j_i=k\\1\le j<k}}\prod_{\ell=1}^{i}\phi_{j_\ell}$$

Since $|\lambda^m\sigma_1| < 1$ by assumption, we can solve for ϕ_k assuming $k \geq m$.

$$\forall k\ge m\qquad \phi_k=\frac{\lambda^k}{1-\lambda^k\sigma_1}\sum_{i=2}^{k}\sigma_i\sum_{\substack{j_1+\cdots+j_i=k\\1\le j<k}}\prod_{\ell=1}^{i}\phi_{j_\ell}$$

This is a recurrence relation where ϕ_k is written in terms of $\{\phi_j\}_{j=1}^{k-1}$. Thus ϕ_k has been specified for all $k \in \mathbb{N}$ and uniqueness has been asserted.

Lemma (2.8)**.** *If $\lambda^k\sigma_1 \neq 1$ for all $k \in \mathbb{Z}^+$, then $\phi(z) = 0$ for all z near the origin (i.e. the trivial solution).*

Under this assumption, the recurrence relation derived in Lemma 2.7 holds for all positive integers. Induction easily proves $\phi_k = 0$ for all $k \in \mathbb{N}$. Thus $\phi(z) = 0$ for all z near the origin.

Lemma (2.9)**.** *We claim $\psi_{n,k} = \psi_{0,k}$ for all $n \in \mathbb{N}$ and $k < m$.*

From the assumption about matching derivatives, we know $\psi_{1,k} = \psi_{0,k}$ for all $k < m$. The formula given in Lemma 2.6 expresses $\psi_{n+1,k}$ as a function Ω_k of $\{\psi_{n,j}\}_{j=1}^{k}$ for all $n, k \in \mathbb{N}$. Since the first m coefficients do not change from ψ_0 to ψ_1, they must never change (Ω_k does not depend on n). Alternatively, it is possible to induct on a chain/product rule symmetry argument similar to parts of the proof from Lemma 2.2.

Recall σ and ψ are holomorphic at the origin. Since power series are absolutely convergent within their radii of convergence, we know $\sum_{i=0}^{\infty}|\sigma_i|\,R_\sigma^i < \infty$ for some $R_\sigma > 0$, which implies $\lim_{i\to\infty}|\sigma_i|\,R_\sigma^i = 0$. Therefore the supremum of $|\sigma_i| R_\sigma^i$ over $i \in \mathbb{N}$ exists and is finite. Define $\alpha := \sup_i |\sigma_i| R_\sigma^i$. Similarly, $\sup_k |\psi_{0,k}| R_\psi^k$ exists and is finite for some $R_\psi > 0$. Without loss of generality, we may assume $R_\sigma < 1$ and $R_\psi < 1$.

Define $\beta > 0$ such that the following conditions are satisfied.

$$|\lambda|^m |\sigma_1| R_\sigma + \alpha |\lambda|^m \big((\beta+1)^{m-1} - 1\big) < R_\sigma \qquad |\lambda|(\beta+1) < \frac{1+|\lambda|}{2}$$

$$\beta\big(\alpha - |\sigma_1| R_\sigma\big) |\lambda|^{m+1} + R_\sigma \left(\frac{1+|\lambda|}{2}\right) < R_\sigma \qquad \beta < 1$$

The conditions on β certainly must seem mysterious, but they have been carefully chosen to help us later in the proof. We justify why $\beta > 0$ can be chosen in such a way to satisfy the conditions. Consider the expressions as functions of β. Replacing β with zero yields:

$$|\lambda|^m |\sigma_1| R_\sigma < R_\sigma \qquad |\lambda| < \frac{1+|\lambda|}{2} \qquad R_\sigma \left(\frac{1+|\lambda|}{2}\right) < R_\sigma \qquad 0 < 1$$

The first statement is implied by the assumption $|\lambda|^m |\sigma'(0)| < 1$. The remaining statements are implied by $0 < |\lambda| < 1$. Increasing β from zero makes the left side of each inequality larger. Making β large could violate the conditions, so we need β sufficiently small. Since the conditions are true when β is replaced with zero, and since the inequalities are strict, continuity of the expressions as functions of β guarantees we can slightly increase to some $\beta > 0$ without violating the conditions. Thus such a choice of β can always be made. Define $\gamma := (\beta R_\psi R_\sigma)^{-1} \left(1 + \sup_k |\psi_{0,k}| R_\psi^k\right)$. Notice $\gamma \geq (\beta R_\psi R_\sigma)^{-1} > 1$.

Lemma (2.10)**.** *We claim the following:*

$$\forall k \geq m \qquad |\lambda|^k |\sigma_1| R_\sigma + \alpha |\lambda|^k \big((\beta+1)^{k-1} - 1\big) \leq R_\sigma$$

Considering $|\lambda| < 1$, $\quad |\sigma_1| R_\sigma \leq \alpha$, and the third condition used in the construction of β, we have the following:

$$\forall k \geq m \qquad \beta\big(\alpha - |\sigma_1| R_\sigma\big) |\lambda|^{k+1} + R_\sigma \left(\frac{1+|\lambda|}{2}\right) \leq R_\sigma$$

Now we are ready to prove the claim by induction. The base case $k = m$ has been assumed in the construction of β. Assuming the statement for some $k \geq m$, we prove it for $k+1$.

$$|\lambda|^k |\sigma_1| R_\sigma + \alpha |\lambda|^k \big((\beta+1)^{k-1} - 1\big) \leq R_\sigma$$

Multiply each side by $|\lambda|(\beta+1)$.

$$|\lambda|^{k+1} |\sigma_1| R_\sigma(\beta+1) + \alpha |\lambda|^{k+1} \big((\beta+1)^k - (\beta+1)\big) \leq R_\sigma |\lambda|(\beta+1) \leq R_\sigma \left(\frac{1+|\lambda|}{2}\right) \implies$$

$$\beta |\lambda|^{k+1} |\sigma_1| R_\sigma - \alpha\beta |\lambda|^{k+1} + |\lambda|^{k+1} |\sigma_1| R_\sigma + \alpha |\lambda|^{k+1} \big((\beta+1)^k - 1\big) \leq R_\sigma \left(\frac{1+|\lambda|}{2}\right) \implies$$

$$|\lambda|^{k+1} |\sigma_1| R_\sigma + \alpha |\lambda|^{k+1} \big((\beta+1)^k - 1\big) \leq \beta\big(\alpha - |\sigma_1| R_\sigma\big) |\lambda|^{k+1} + R_\sigma \left(\frac{1+|\lambda|}{2}\right) \leq R_\sigma$$

Thus the claim is proven.

Lemma (2.11)**.** *We claim $|\psi_{n,k}| \leq R_\sigma \beta \gamma^k$ for all $n, k \in \mathbb{N}$.*

We use induction. First, we prove the base case $|\psi_{0,k}| \leq R_\sigma \beta \gamma^k$ for all $k \in \mathbb{N}$. Since $\psi_{0,0} = 0$, it suffices to consider only $k \geq 1$. We use $(R_\sigma \beta)^{1-k} \geq 1$ for all $k \geq 1$ which follows from $R_\sigma \beta < \beta < 1$.

$$\forall k \geq 1 \qquad |\psi_{0,k}| \leq R_\psi^{-k} \sup_{j \in \mathbb{N}} |\psi_{0,j}| R_\psi^j \leq R_\psi^{-k} \left(1 + \sup_{j \in \mathbb{N}} |\psi_{0,j}| R_\psi^j\right)^k \leq R_\sigma \beta (\beta R_\psi R_\sigma)^{-k} \left(1 + \sup_{j \in \mathbb{N}} |\psi_{0,j}| R_\psi^j\right)^k = R_\sigma \beta \gamma^k$$

With the base case verified, we prove the inductive step. Assuming $|\psi_{n,k}| \le R_\sigma \beta \gamma^k$ for all $k \in \mathbb{N}$, we prove $|\psi_{n+1,\,k}| \le R_\sigma \beta \gamma^k$ for all $k \in \mathbb{N}$. Due to Lemma 2.9, we only need to check the condition for $k \ge m$. We use Lemma 2.6.

$$\forall k \ge m \qquad |\psi_{n+1,k}| = \left| \lambda^k \sum_{i=1}^{k} \sigma_i \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} \prod_{\ell=1}^{i} \psi_{n,\,j_\ell} \right| \le |\lambda|^k \sum_{i=1}^{k} |\sigma_i| \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} \prod_{\ell=1}^{i} |\psi_{n,\,j_\ell}| \le$$

Use the inductive assumption.

$$|\lambda|^k \sum_{i=1}^{k} |\sigma_i| \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} \prod_{\ell=1}^{i} R_\sigma \beta \gamma^{j_\ell} =$$

$$|\lambda|^k \gamma^k \sum_{i=1}^{k} |\sigma_i| \, R_\sigma^i \beta^i \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} 1 =$$

Apply Lemma 2.4

$$|\lambda|^k \gamma^k \sum_{i=1}^{k} \binom{k-1}{i-1} |\sigma_i| \, R_\sigma^i \beta^i =$$

$$|\lambda|^k \gamma^k |\sigma_1| \, R_\sigma \beta + |\lambda|^k \gamma^k \sum_{i=2}^{k} \binom{k-1}{i-1} |\sigma_i| \, R_\sigma^i \beta^i \le$$

Use the definition of $\alpha = \sup_i |\sigma_i| \, R_\sigma^i$.

$$|\lambda|^k \gamma^k |\sigma_1| \, R_\sigma \beta + |\lambda|^k \gamma^k \alpha \sum_{i=2}^{k} \binom{k-1}{i-1} \beta^i =$$

Decrement the summation index.

$$|\lambda|^k \gamma^k |\sigma_1| \, R_\sigma \beta + |\lambda|^k \gamma^k \alpha \beta \sum_{i=1}^{k-1} \binom{k-1}{i} \beta^i =$$

$$\beta \gamma^k \left(|\lambda|^k |\sigma_1| \, R_\sigma + |\lambda|^k \alpha \sum_{i=1}^{k-1} \binom{k-1}{i} \beta^i \right) =$$

Apply the binomial theorem and then Lemma 2.10 to yield the desired result.

$$\beta \gamma^k \left(|\lambda|^k |\sigma_1| \, R_\sigma + \alpha \, |\lambda|^k \left((\beta+1)^{k-1} - 1 \right) \right) \le R_\sigma \beta \gamma^k$$

Thus the lemma is proven.

Multiplying Lemma 2.11 by γ^{-2k}, we have $|\psi_{n,k}| \, \gamma^{-2k} \le R_\sigma \beta \gamma^{-k}$ for all $n, k \in \mathbb{N}$. This implies ψ_n is holomorphic on a disc of radius γ^{-2} centered at the origin for all $n \in \mathbb{N}$. It suffices to show the power series for ψ_n is absolutely convergent at a radius of γ^{-2}. Recall $\gamma > 1$, which means $0 < \gamma^{-1} < 1$.

$$\forall n \in \mathbb{N} \qquad \sum_{k=0}^{\infty} |\psi_{n,k}| \, (\gamma^{-2})^k \le \sum_{k=0}^{\infty} R_\sigma \beta (\gamma^{-1})^k = \frac{R_\sigma \beta}{1-\gamma^{-1}} < \infty$$

We are almost ready to put everything together; just a few more definitions. Define $\omega(x) := |\lambda|^m \sum_{i=1}^{\infty} i \, |\sigma_i| \, x^{i-1}$ for sufficiently small $x \in \mathbb{R}$. ω is related to the derivative of σ. Since σ' is holomorphic at the origin, ω is well-defined and continuous in some neighborhood of $x = 0$. Choose $\delta > 0$ such that the following conditions are satisfied.

$$\omega(\delta) < 1 \qquad\qquad \frac{\delta}{\gamma \, |\lambda| \, (R_\sigma \beta + \delta)} < \gamma^{-2}$$

We justify our choice of δ similarly to how we justified our choice of β. Replacing δ with zero yields $|\lambda|^m |\sigma_1| < 1$ and $0 < \gamma^{-2}$, both of which are true. If δ is increased from zero, the left side of each inequality is increased. By continuity of the expressions as functions of δ, such a choice of δ can always be made. Define $r := \frac{\delta}{\gamma|\lambda|(R_\sigma\beta+\delta)}$. We have $0 < r\gamma|\lambda| < 1$ and $0 < r < \gamma^{-2}$.

Define the metric space $\mathcal{S} := \{ \psi_n \mid n \in \mathbb{N} \}$. Define the norm $\|f\| := \sum_{k=0}^{\infty} |f_k| r^k$ on functions f holomorphic at the origin where f_k is the k^{th} coefficient of the power series expansion. Since $r < \gamma^{-2}$, we must have $\|\psi_n\| < \infty$ for all $n \in \mathbb{N}$. This norm induces a metric $d(f,g) := \|f - g\|$ on $\mathcal{S}$. We intend to show L is a contraction on $\mathcal{S}$ with Lipschitz constant $\omega(\delta)$.

$$\forall f,g \in \mathcal{S} \qquad d\big(Lf,\ Lg\big) = \|Lf - Lg\| = \sum_{k=0}^{\infty} |(Lf)_k - (Lg)_k|\, r^k =$$

Apply Lemma 2.9

$$\sum_{k=m}^{\infty} |(Lf)_k - (Lg)_k|\, r^k =$$

Apply Lemma 2.6

$$\sum_{k=m}^{\infty} \left| \lambda^k \sum_{i=1}^{k} \sigma_i \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} \left(\prod_{\ell=1}^{i} f_{j_\ell} - \prod_{\ell=1}^{i} g_{j_\ell} \right) \right| r^k =$$

Express the difference of products as a telescoping sum.

$$\sum_{k=m}^{\infty} \left| \lambda^k \sum_{i=1}^{k} \sigma_i \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} \sum_{\ell=1}^{i} g_{j_1}\cdots g_{j_{\ell-1}} (f_{j_\ell} - g_{j_\ell}) f_{j_{\ell+1}} \cdots f_{j_i} \right| r^k \le$$

Use the triangle inequality.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \sum_{i=1}^{k} |\sigma_i| \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} \sum_{\ell=1}^{i} |g_{j_1}|\cdots |g_{j_{\ell-1}}|\, |f_{j_\ell} - g_{j_\ell}|\, |f_{j_{\ell+1}}| \cdots |f_{j_i}| \le$$

Apply Lemma 2.11

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \sum_{i=1}^{k} |\sigma_i|\, R_\sigma^{i-1}\beta^{i-1} \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} \sum_{\ell=1}^{i} \gamma^{j_1+\cdots+j_{\ell-1}}\, |f_{j_\ell} - g_{j_\ell}|\, \gamma^{j_{\ell+1}+\cdots+j_i} =$$

Notice $(j_1 + \cdots + j_{\ell-1}) + (j_{\ell+1} + \cdots + j_i) = k - j_\ell$.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \sum_{i=1}^{k} |\sigma_i|\, R_\sigma^{i-1}\beta^{i-1} \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} \sum_{\ell=1}^{i} |f_{j_\ell} - g_{j_\ell}|\, \gamma^{k-j_\ell} =$$

Switch the j,ℓ summations.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \sum_{i=1}^{k} |\sigma_i|\, R_\sigma^{i-1}\beta^{i-1} \sum_{\ell=1}^{i} \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} |f_{j_\ell} - g_{j_\ell}|\, \gamma^{k-j_\ell} =$$

Considering symmetry, the j sum is independent of ℓ.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \sum_{i=1}^{k} i\, |\sigma_i|\, R_\sigma^{i-1}\beta^{i-1} \sum_{\substack{j_1+\cdots+j_i=k \\ 1\le j\le k}} |f_{j_1} - g_{j_1}|\, \gamma^{k-j_1} =$$

Split into cases $i = 1$ and $i > 1$.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \left(|\sigma_1| \, |f_k - g_k| + \sum_{i=2}^{k} i \, |\sigma_i| \, R_\sigma^{i-1} \beta^{i-1} \sum_{\substack{j_1+\cdots+j_i=k \\ 1 \le j \le k}} |f_{j_1} - g_{j_1}| \, \gamma^{k-j_1} \right) =$$

Fixing $j := j_1$, there are $\binom{k-j_1-1}{i-2}$ ways for $(j_2, \ldots, j_i)$ to sum to $k - j_1$ applying Lemma 2.4

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \left(|\sigma_1| \, |f_k - g_k| + \sum_{i=2}^{k} i \, |\sigma_i| \, R_\sigma^{i-1} \beta^{i-1} \sum_{j=1}^{k-i+1} \binom{k-j-1}{i-2} |f_j - g_j| \, \gamma^{k-j} \right) =$$

Since everything is non-negative, we may rearrange terms however we like.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \, |\sigma_1| \, |f_k - g_k| + \sum_{k=m}^{\infty} \sum_{i=2}^{k} \sum_{j=1}^{k-i+1} \binom{k-j-1}{i-2} |f_j - g_j| \, i \, |\sigma_i| \, R_\sigma^{i-1} \beta^{i-1} \, |\lambda|^k \, r^k \gamma^{k-j} =$$

Consider the terms of the triple summation. By Lemma 2.9, the terms are zero when $j < m$. Since $m \le j \le k-i+1$ for non-zero terms, we have $i \le k - m + 1$, which is better than the bound we already have $i \le k$. If $k = m$, the middle sum is over $2 \le i \le 1$ using the new bound, which evaluates to zero, so we can start with $k = m + 1$.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \, |\sigma_1| \, |f_k - g_k| + \sum_{k=m+1}^{\infty} \sum_{i=2}^{k-m+1} \sum_{j=m}^{k-i+1} \binom{k-j-1}{i-2} |f_j - g_j| \, i \, |\sigma_i| \, R_\sigma^{i-1} \beta^{i-1} \, |\lambda|^k \, r^k \gamma^{k-j} =$$

We interchange the summation order $k, i, j \to i, j, k$.

$$\sum_{k=m+1}^{\infty} \sum_{i=2}^{k-m+1} \sum_{j=m}^{k-i+1} = \sum_{i=2}^{\infty} \sum_{k=m+i-1}^{\infty} \sum_{j=m}^{k-i+1} = \sum_{i=2}^{\infty} \sum_{j=m}^{\infty} \sum_{k=j+i-1}^{\infty}$$

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \, |\sigma_1| \, |f_k - g_k| + \sum_{i=2}^{\infty} \sum_{j=m}^{\infty} \sum_{k=j+i-1}^{\infty} \binom{k-j-1}{i-2} |f_j - g_j| \, i \, |\sigma_i| \, R_\sigma^{i-1} \beta^{i-1} \, |\lambda|^k \, r^k \gamma^{k-j} =$$

Shift the k summation index.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \, |\sigma_1| \, |f_k - g_k| + \sum_{i=2}^{\infty} \sum_{j=m}^{\infty} \sum_{k=0}^{\infty} \binom{k+i-2}{i-2} |f_j - g_j| \, i \, |\sigma_i| \, R_\sigma^{i-1} \beta^{i-1} \, |\lambda|^{k+j+i-1} \, r^{k+j+i-1} \gamma^{k+i-1} =$$

Pull factors independent of k out of the innermost summation.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \, |\sigma_1| \, |f_k - g_k| + \sum_{i=2}^{\infty} \sum_{j=m}^{\infty} |f_j - g_j| \, i \, |\sigma_i| \, R_\sigma^{i-1} \beta^{i-1} \, |\lambda|^{j+i-1} \, r^{j+i-1} \gamma^{i-1} \sum_{k=0}^{\infty} \binom{k+i-2}{i-2} (r\gamma \, |\lambda|)^k =$$

Apply Lemma 2.5 using $0 < r\gamma \, |\lambda| < 1$.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \, |\sigma_1| \, |f_k - g_k| + \sum_{i=2}^{\infty} \sum_{j=m}^{\infty} |f_j - g_j| \, i \, |\sigma_i| \, R_\sigma^{i-1} \beta^{i-1} \, |\lambda|^{j+i-1} \, r^{j+i-1} \gamma^{i-1} \left(\frac{1}{1 - r\gamma \, |\lambda|} \right)^{i-1} =$$

Bring the factors with powers of $i - 1$ together.

$$\sum_{k=m}^{\infty} |\lambda|^k r^k \, |\sigma_1| \, |f_k - g_k| + \sum_{i=2}^{\infty} \sum_{j=m}^{\infty} |f_j - g_j| \, i \, |\sigma_i| \, (r \, |\lambda|)^j \left(\frac{R_\sigma \beta r \gamma \, |\lambda|}{1 - r\gamma \, |\lambda|} \right)^{i-1} =$$

Separate the i, j sums on the right, and rewrite the left sum.

$$|\sigma_1| \sum_{j=m}^{\infty} |f_j - g_j| \, (r \, |\lambda|)^j + \left(\sum_{i=2}^{\infty} i \, |\sigma_i| \left(\frac{R_\sigma \beta r \gamma \, |\lambda|}{1 - r\gamma \, |\lambda|} \right)^{i-1} \right) \left(\sum_{j=m}^{\infty} |f_j - g_j| \, (r \, |\lambda|)^j \right) =$$

Bring the left sum into the right sum as the case $i = 1$.

$$\left(\sum_{i=1}^{\infty} i \, |\sigma_i| \left(\frac{R_\sigma \beta r \gamma \, |\lambda|}{1 - r\gamma \, |\lambda|} \right)^{i-1} \right) \left(\sum_{j=m}^{\infty} |f_j - g_j| \, (r \, |\lambda|)^j \right) =$$

Move $|\lambda|^m$ from the right sum to the left.

$$\left(|\lambda|^m \sum_{i=1}^{\infty} i\,|\sigma_i| \left(\frac{R_\sigma \beta r\gamma\,|\lambda|}{1-r\gamma\,|\lambda|}\right)^{i-1}\right)\left(\sum_{j=m}^{\infty} |f_j - g_j|\, r^j\, |\lambda|^{j-m}\right) =$$

Recall the definition of ω.

$$\omega\left(\frac{R_\sigma \beta r\gamma\,|\lambda|}{1-r\gamma\,|\lambda|}\right)\left(\sum_{j=m}^{\infty} |f_j - g_j|\, r^j\, |\lambda|^{j-m}\right)$$

Simplify the argument of ω using the definition of r.

$$\frac{R_\sigma \beta r\gamma\,|\lambda|}{1-r\gamma\,|\lambda|} = \frac{R_\sigma \beta \left(\frac{\delta}{\gamma|\lambda|(R_\sigma\beta+\delta)}\right)\gamma\,|\lambda|}{1-\left(\frac{\delta}{\gamma|\lambda|(R_\sigma\beta+\delta)}\right)\gamma\,|\lambda|} = \frac{R_\sigma\beta\delta\gamma\,|\lambda|}{\gamma\,|\lambda|\,(R_\sigma\beta+\delta)-\delta\gamma\,|\lambda|} = \frac{R_\sigma\beta\delta\gamma\,|\lambda|}{\gamma\,|\lambda|\,R_\sigma\beta} = \delta$$

$$\omega(\delta)\sum_{j=m}^{\infty} |f_j - g_j|\, r^j\, |\lambda|^{j-m} \le$$

Since $|\lambda| < 1$, we have $|\lambda|^{j-m} \le 1$ for all $j \ge m$.

$$\omega(\delta)\sum_{j=m}^{\infty} |f_j - g_j|\, r^j =$$

Recall the definition of $d(f,g)$. Lemma 2.9 may be used to assert equality.

$$\omega(\delta)d(f,g)$$

We have shown $d\big(Lf,\ Lg\big) \le \omega(\delta)d(f,g)$ for all $f, g \in \mathcal{S}$. By construction, $\omega(\delta) < 1$, so L is a contraction mapping on $\mathcal{S} = \{\ \psi_n \mid n \in \mathbb{N}\ \}$. This implies the sequence $\{\psi_n\}_{n=0}^{\infty}$ is Cauchy with respect to the metric (refer to a proof of the Banach fixed point theorem [13, Theorem 9.23]). Let us write out what this means. For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{k=0}^{\infty} |\psi_{i,k} - \psi_{j,k}|\, r^k < \epsilon$ for all $i, j \ge N$.

We claim $\{\psi_n\}_{n=0}^{\infty}$ is uniformly Cauchy on the open disc $\mathcal{D}$ of radius r centered at the origin. Take any $\epsilon > 0$. Take the N at ϵ from the definition of $\{\psi_n\}_{n=0}^{\infty}$ being Cauchy in the coefficient metric. Take any $i, j \ge N$ and $z \in \mathbb{C}$ such that $|z| < r$.

$$|\psi_i(z) - \psi_j(z)| = \left|\sum_{k=0}^{\infty}(\psi_{i,k} - \psi_{j,k})z^k\right| \le \sum_{k=0}^{\infty} |\psi_{i,k} - \psi_{j,k}|\,|z|^k \le \sum_{k=0}^{\infty} |\psi_{i,k} - \psi_{j,k}|\, r^k < \epsilon$$

Thus our claim is proven. Define $\phi(z) := \lim_{n\to\infty} \psi_n(z)$ for all $z \in \mathcal{D}$ (we've shown this limit exists and is uniform). Since ϕ is a uniform limit of functions which are holomorphic on $\mathcal{D}$, ϕ is holomorphic on $\mathcal{D}$. As a consequence of Lemma 2.9, we have $\phi^{(k)}(0) = \psi^{(k)}(0)$ for all $k < m$. We would like to prove $\phi(z) = \sigma \circ \phi(\lambda z)$ for all $z \in \mathcal{D}$, but first we need a lemma.

Lemma (2.12)**.** *We claim $|\phi(\lambda z)| \le R_\sigma$ for all $z \in \mathcal{D}$.*

By the definition of ϕ, it suffices to show $|\psi_n(\lambda z)| \le R_\sigma$ for all $z \in \mathcal{D}$ and $n \in \mathbb{N}$. Recall the radius of r of $\mathcal{D}$ is less than γ^{-2}. We also use Lemma 2.11.

$$\forall z \in \mathcal{D} \qquad |\psi_n(\lambda z)| = \left|\sum_{k=1}^{\infty} \psi_{n,k}\lambda^k z^k\right| \le \sum_{k=1}^{\infty} |\psi_{n,k}|\,|\lambda|^k\,|z|^k \le \sum_{k=1}^{\infty} |\psi_{n,k}|\,|\lambda|^k\,\gamma^{-2k} \le R_\sigma\beta \sum_{k=1}^{\infty} |\lambda|^k\,\gamma^{-k} =$$

Since $|\lambda| < 1 < \gamma$, we may sum the geometric series.

$$\frac{R_\sigma\beta\,|\lambda|\,\gamma^{-1}}{1-|\lambda|\,\gamma^{-1}} = \frac{R_\sigma\beta\,|\lambda|}{\gamma - |\lambda|} \le R_\sigma$$

The last step is justified by the second condition in the definition of β.

$$|\lambda|(\beta+1)<1<\gamma \implies \beta|\lambda| \le \gamma-|\lambda| \implies \frac{\beta|\lambda|}{\gamma-|\lambda|} \le 1$$

Thus the lemma is proven.

The previous lemma implies σ is holomorphic at $\phi(\lambda z)$ for all $z \in \mathcal{D}$ (and hence is continuous at $\phi(\lambda z)$). We are ready to prove $\phi(z) = \sigma \circ \phi(\lambda z)$ for all $z \in \mathcal{D}$. We use continuity of σ to move the limit defining ϕ.

$$\forall z \in \mathcal{D} \qquad \sigma \circ \phi(\lambda z) = \sigma\left(\lim_{n\to\infty} \psi_n(\lambda z)\right) = \lim_{n\to\infty} \sigma \circ \psi_n(\lambda z) = \lim_{n\to\infty} \psi_{n+1}(z) = \phi(z)$$

We have completed the proof of Theorem 2.3, and hence Theorem 2. We intend to provide theorems extending the region of convergence $\mathcal{D}$ given additional assumptions. First, we prove a lemma.

Lemma (2.13)**.** *Suppose we are given a compact set $K \subset \mathbb{C}$, an open set $U \subset \mathbb{C}$, continuous functions $f : K \to \mathbb{C}$ and $g : U \to \mathbb{C}$ where $f(K) \subset U$, and a sequence of functions $\{f_n : K \to \mathbb{C}\}_{n=0}^{\infty}$ such that $f(z) = \lim_{n\to\infty} f_n(z)$ uniformly on K. Then $g \circ f(z) = \lim_{n\to\infty} g \circ f_n(z)$ uniformly on K.*

Since K is compact and f is continuous, $f(K)$ is compact. Define the closed set $\Omega := \mathbb{C} \setminus U$. Since $f(K) \subset U$, it must be that $f(K)$ and Ω are disjoint. In a metric space, the distance between disjoint closed and compacts sets is positive [13, Chapter 4, Exercise 21]. So there exists $t > 0$ such that $d(x, y) > t$ for all $x \in K$ and $y \in \Omega$. For any $r > 0$ and $z_0 \in \mathbb{C}$, let $D_r(z_0) := \{\ z \in \mathbb{C} \mid |z - z_0| \le r\ \}$ denote the closed disc of radius r centered at z_0.

Since $f(K)$ is compact, there exists $r > 0$ such that $f(K) \subset D_r(0)$. Define $L := D_{r+t}(0) \cap \bigcap_{z\in\Omega} \{\ w \in \mathbb{C} \mid |w - z| \ge t/2\ \}$ (when $\Omega = \emptyset$, the empty intersection is equal to $\mathbb{C}$ by definition). From $D_{r+t}(0)$, we are essentially removing all open balls of radius $t/2$ with centers in Ω, so $L \subset U$. Since Ω and $f(K)$ are separated by a distance t, it follows that $f(K) \subset L$. Furthermore, $D_{t/2}(z) \subset L$ for all $z \in f(K)$ (this will be important later). Since L is an intersection of closed sets, L is closed. Since L is closed and bounded, L is compact. Since $L \subset U$ is compact and g is continuous, g is uniformly continuous on L.

Hence, the following statements are true.

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in L \qquad |x - y| < \delta \implies |g(x) - g(y)| < \epsilon$$

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall z \in K \quad \forall n \ge N \qquad |f_n(z) - f(z)| < \epsilon$$

And we want to prove:

$$\forall \epsilon > 0 \quad \exists M \in \mathbb{N} \quad \forall z \in K \quad \forall n \ge M \qquad |g(f_n(z)) - g(f(z))| < \epsilon$$

M, N, and δ may be considered as functions of ϵ. We claim $M[\epsilon] := N\big[\min(t/2,\ \delta[\epsilon])\big]$ works to prove the desired statement. Take any $\epsilon > 0$, $z \in K$, and $n \ge M[\epsilon]$. Then $|f_n(z) - f(z)| < \min(t/2,\ \delta[\epsilon])$ by the definition of N. Since $D_{t/2}(f(z)) \subset L$, it must be that $f(z)$ and $f_n(z)$ are in L. Then $|g(f_n(z)) - g(f(z))| < \epsilon$ by the definition of δ. Thus $g \circ f(z) = \lim_{n\to\infty} g \circ f_n(z)$ where the limit occurs uniformly on K as desired.

Theorem (2.14)**.** *Suppose ψ is holomorphic at $z_0 \in \mathbb{C}$. Suppose τ is holomorphic on an open set $U \subset \mathbb{C}$ containing z_0. Suppose σ is an entire function. Let $w_0 := \psi(z_0)$ and $\lambda := \tau'(z_0)$. Suppose $\tau(U) \subset U$, $\tau(z_0) = z_0$, $\sigma(w_0) = w_0$, and $0 < |\lambda| < 1$. Suppose $\lim_{n\to\infty} \tau^{\circ n}(z) = z_0$ uniformly on compacts $K \subset U$. Define $\psi_n = \sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ for all $n \in \mathbb{N}$. Suppose there exists $m \in \mathbb{Z}^+$ such that $|\lambda|^m |\sigma'(w_0)| < 1$ and $\psi_0^{(k)}(z_0) = \psi_1^{(k)}(z_0)$ for all $k < m$.*

Then $\{\psi_n\}_{n=0}^{\infty}$ converges uniformly on compacts $K \subset U$ to a holomorphic function ϕ. Furthermore, ϕ is the unique function on U which is holomorphic at z_0 and satisfies $\phi = \sigma \circ \phi \circ \tau$ and $\phi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$ (including $k = 0$).

Invoke Theorem 2 to get an open disc $\mathcal{D}$ of positive radius centered at z_0 such that $\tau(\mathcal{D}) \subset \mathcal{D}$ where $\{\psi_n\}_{n=0}^{\infty}$ converges uniformly on $\mathcal{D}$ to a holomorphic function φ. Furthermore, φ is the unique holomorphic function on $\mathcal{D}$ satisfying $\varphi = \sigma \circ \varphi \circ \tau$ and $\varphi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$. WLOG we may assume $\mathcal{D} \subset U$ (shrink the radius of $\mathcal{D}$ if necessary).

First, we prove uniqueness. Suppose ϕ is a function on U which is holomorphic at z_0 and satisfies $\phi = \sigma \circ \phi \circ \tau$ and $\phi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$. Then $\phi|_{\mathcal{D}} = \varphi$ by uniqueness of φ (shrink $\mathcal{D}$ again if necessary). Take any $z \in U$. Since $\lim_{n\to\infty} \tau^{\circ n}(z) = z_0$, we can find $j \in \mathbb{N}$ such that $\tau^{\circ j}(z) \in \mathcal{D}$. Then $\phi(z) = \sigma^{\circ j} \circ \phi \circ \tau^{\circ j}(z) = \sigma^{\circ j} \circ \varphi \circ \tau^{\circ j}(z)$, so $\phi(z)$ is uniquely specified for every $z \in U$.

Now we prove convergence. Take any compact set $K \subset U$. Since $\tau^{\circ n}$ converges uniformly to z_0 on K, we can find $j \in \mathbb{N}$ such that $\tau^{\circ j}(K) \subset \mathcal{D}$. Define a function $\phi_K : K \to \mathbb{C}$ where $\phi_K := \sigma^{\circ j} \circ \varphi \circ \tau^{\circ j}$ (which is well-defined, even though φ is only defined on $\mathcal{D}$). By the chain rule, ϕ_K is holomorphic on the interior of K. We know $\varphi \circ \tau^{\circ j}(z) = \lim_{n\to\infty} \psi_n \circ \tau^{\circ j}(z)$ for all $z \in K$ where the limit occurs uniformly on K because $\tau^{\circ j}(K) \subset \mathcal{D}$. Invoke Lemma 2.13.

$$\forall z \in K \qquad \phi_K(z) = \sigma^{\circ j} \circ \varphi \circ \tau^{\circ j}(z) = \sigma^{\circ j}\left(\lim_{n\to\infty} \psi_n \circ \tau^{\circ j}(z)\right) = \lim_{n\to\infty} \sigma^{\circ j} \circ \psi_n \circ \tau^{\circ j}(z) = \lim_{n\to\infty} \psi_{n+j}(z) = \lim_{n\to\infty} \psi_n(z)$$

So $\phi_K(z) = \lim_{n\to\infty} \psi_n(z)$ uniformly on K. Define $\phi(z) := \lim_{n\to\infty} \psi_n(z)$ for all $z \in U$ (which we've shown exists). Take any $z \in U$. Since U is open there exists an open disc of radius $r > 0$ centered at z and contained by U. Let K be the closed disc of radius $r/2$ centered at z. K is compact, so ϕ_K is holomorphic at z (which is in the interior of K). Since $\phi|_K = \phi_K$, ϕ is also holomorphic at z. Thus ϕ is holomorphic on U. Since $\phi|_{\mathcal{D}} = \varphi$ by their common definition as the ψ_n limit, it must be that $\phi^{(k)}(z_0) = \varphi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$. Continuity of σ shows $\phi = \sigma \circ \phi \circ \tau$ (we are also using $\tau(U) \subset U$).

$$\forall z \in U \qquad \sigma \circ \phi \circ \tau(z) = \sigma\left(\lim_{n\to\infty} \psi_n \circ \tau(z)\right) = \lim_{n\to\infty} \sigma \circ \psi_n \circ \tau(z) = \lim_{n\to\infty} \psi_{n+1}(z) = \phi(z)$$

Hence Theorem 2.14 is proven.

Theorem (2.15)**.** *Suppose ψ is holomorphic at $z_0 \in \mathbb{C}$. Suppose τ is holomorphic on an open set $U \subset \mathbb{C}$ containing z_0. Suppose σ is a rational function (quotient of polynomials). Let $w_0 := \psi(z_0)$ and $\lambda := \tau'(z_0)$. Suppose $\tau(U) \subset U$, $\tau(z_0) = z_0$, $\sigma(w_0) = w_0$, and $0 < |\lambda| < 1$. Suppose $\lim_{n\to\infty} \tau^{\circ n}(z) = z_0$ uniformly on compacts $K \subset U$. Define $\psi_n = \sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ for all $n \in \mathbb{N}$. Suppose there exists $m \in \mathbb{Z}^+$ such that $|\lambda|^m |\sigma'(w_0)| < 1$ and $\psi_0^{(k)}(z_0) = \psi_1^{(k)}(z_0)$ for all $k < m$.*

Then $\{\psi_n\}_{n=0}^{\infty}$ has pointwise convergence to a meromorphic function ϕ on U and the limit occurs uniformly on all compacts $K \subset U$ not containing singularities of ϕ. Furthermore, ϕ is the unique function $U \to \mathbb{C}^*$ which is holomorphic at z_0 and satisfies $\phi = \sigma \circ \phi \circ \tau$ and $\phi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$ (including $k = 0$).

(Remark) '$\phi(z) = \sigma \circ \phi \circ \tau(z)$ for all $z \in U$' only makes sense when we consider the extended complex plane $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ with $\tau : U \to U$, $\phi : U \to \mathbb{C}^*$, and $\sigma : \mathbb{C}^* \to \mathbb{C}^*$. The topology on $\mathbb{C}^*$ is given by the one-point compactification of the usual topology on $\mathbb{C}$. From a quotient of polynomials P/Q (with Q not identically zero) inherently only defined on $\mathbb{C} \setminus \Omega$ for some finite set $\Omega \subset \mathbb{C}$, we define σ to be the unique continuous extension of P/Q onto all of $\mathbb{C}^*$. In other words, we specify $\sigma(z) = \lim_{w\to z} \sigma(w)$ for all $z \in \mathbb{C}^*$ (c.f. the remark beneath Theorem 3.5). Such an extension always exists for P/Q.

$\sigma(w_0) = w_0$ implies σ is holomorphic at w_0, since rational functions are differentiable away from poles. Invoke Theorem 2 to get an open disc $\mathcal{D}$ of positive radius centered at z_0 such that $\tau(\mathcal{D}) \subset \mathcal{D}$ where $\{\psi_n\}_{n=0}^{\infty}$ converges uniformly on $\mathcal{D}$ to a holomorphic function φ. Furthermore, φ is the unique holomorphic function on $\mathcal{D}$ satisfying $\varphi = \sigma \circ \varphi \circ \tau$ and $\varphi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$. WLOG we may assume $\mathcal{D} \subset U$ (shrink the radius of $\mathcal{D}$ if necessary).

We show uniqueness. Suppose ϕ is a function $U \to \mathbb{C}^*$ which is holomorphic at z_0 and satisfies $\phi = \sigma \circ \phi \circ \tau$ and $\phi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$. Then $\phi|_{\mathcal{D}} = \varphi$ by uniqueness of φ (shrink $\mathcal{D}$ again if necessary). Take any $z \in U$. Since $\lim_{n\to\infty} \tau^{\circ n}(z) = z_0$, we can find $j \in \mathbb{N}$ such that $\tau^{\circ j}(z) \in \mathcal{D}$. Then $\phi(z) = \sigma^{\circ j} \circ \phi \circ \tau^{\circ j}(z) = \sigma^{\circ j} \circ \varphi \circ \tau^{\circ j}(z)$, so $\phi(z)$ is uniquely specified for every $z \in U$.

We show pointwise convergence. Take any $z \in U$. We can find $j \in \mathbb{N}$ such that $\tau^{\circ j}(z) \in \mathcal{D}$. Use continuity of $\sigma^{\circ j}$ on $\mathbb{C}^*$.

$$\sigma^{\circ j} \circ \varphi \circ \tau^{\circ j}(z) = \sigma^{\circ j}\left(\lim_{n\to\infty} \psi_n \circ \tau^{\circ j}(z)\right) = \lim_{n\to\infty} \sigma^{\circ j} \circ \psi_n \circ \tau^{\circ j} = \lim_{n\to\infty} \psi_{n+j}(z) = \lim_{n\to\infty} \psi_n(z)$$

Thus the pointwise limit of exists on U, and we may define $\phi(z) := \lim_{n\to\infty} \psi_n(z)$ for all $z \in U$. Since $\phi|_{\mathcal{D}} = \varphi$ by their common definition as the ψ_n limit, it must be that $\phi^{(k)}(z_0) = \varphi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$. Continuity of σ on $\mathbb{C}^*$ shows $\phi = \sigma \circ \phi \circ \tau$.

$$\forall z \in U \qquad \sigma \circ \phi \circ \tau(z) = \sigma\left(\lim_{n\to\infty} \psi_n \circ \tau(z)\right) = \lim_{n\to\infty} \sigma \circ \psi_n \circ \tau(z) = \lim_{n\to\infty} \psi_{n+1}(z) = \phi(z)$$

We show ϕ is a meromorphic function. Take any compact set $K \subset U$. Since $\tau^{\circ n}$ converges uniformly to z_0 on K, we can find $j \in \mathbb{N}$ such that $\tau^{\circ j}(K) \subset \mathcal{D}$. Then $\phi(z) = \sigma^{\circ j} \circ \phi \circ \tau^{\circ j}(z) = \sigma^{\circ j} \circ \varphi \circ \tau^{\circ j}(z)$ for all $z \in K$. The expression $\sigma^{\circ j} \circ \varphi \circ \tau^{\circ j}$ is meromorphic on the interior of K since it is a rational function $\sigma^{\circ j}$ composed with a holomorphic function $\varphi \circ \tau^{\circ j}$. So ϕ is meromorphic on the interior of every compact set $K \subset U$. It follows that ϕ is meromorphic on all of U.

We show the desired form of uniform convergence. Take any compact set $K \subset U$ such that $K \cap \phi^{-1}(\{\infty\}) = \emptyset$ (i.e. avoiding the singularities of ϕ). We can find $j \in \mathbb{N}$ such that $\tau^{\circ j}(K) \subset \mathcal{D}$. Since $\phi(z) = \sigma^{\circ j} \circ \varphi \circ \tau^{\circ j}(z)$ for all $z \in K$, it follows that $\sigma^{\circ j}$ does not have any singularities in the image $\varphi \circ \tau^{\circ j}(K)$ (otherwise K would contain a singularity of ϕ). We know $\varphi \circ \tau^{\circ j}(z) = \lim_{n\to\infty} \psi_n \circ \tau^{\circ j}(z)$ for all $z \in K$ where the limit occurs uniformly on K because $\tau^{\circ j}(K) \subset \mathcal{D}$. Invoke Lemma 2.13 considering $\sigma^{\circ j}$ as a continuous function on $\mathbb{C}$ minus singularities.

$$\forall z \in K \qquad \phi(z) = \sigma^{\circ j} \circ \varphi \circ \tau^{\circ j}(z) = \sigma^{\circ j}\left(\lim_{n\to\infty} \psi_n \circ \tau^{\circ j}(z)\right) = \lim_{n\to\infty} \sigma^{\circ j} \circ \psi_n \circ \tau^{\circ j}(z) = \lim_{n\to\infty} \psi_{n+j}(z) = \lim_{n\to\infty} \psi_n(z)$$

So $\phi(z) = \lim_{n\to\infty} \psi_n(z)$ uniformly on K. Thus Theorem 2.15 is proven.

Consequences

Corollary (3.1)**.** *Suppose ϕ and τ are holomorphic at $z_0 \in \mathbb{C}$. Suppose σ is holomorphic at $w_0 := \phi(z_0)$. Let $\lambda := \tau'(z_0)$. Suppose $\tau(z_0) = z_0$, $\sigma(w_0) = w_0$, and $0 < |\lambda| < 1$. Additionally suppose $\phi = \sigma \circ \phi \circ \tau$ in some neighborhood of z_0.*

Then there exists composition expansions for ϕ about z_0, which means: Take the smallest $m \in \mathbb{Z}^+$ such that $|\lambda|^m |\sigma'(w_0)| < 1$. Take any ψ holomorphic at z_0 such that $\phi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$ (including $k = 0$). Then there exists a neighborhood $\mathcal{D}$ of z_0 (depending on ψ) such that $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ uniformly converges to ϕ on $\mathcal{D}$ as $n \to \infty$.

This is an immediate consequence of Theorem 2. There are variants of this corollary which follow from Theorem 2.14 and Theorem 2.15. As is required to invoke our theorems, we explain why the k^{th} derivatives of ψ and $\sigma \circ \psi \circ \tau$ evaluated at z_0 are equal for all $k < m$. It essentially follows from the assumptions $\phi = \sigma \circ \phi \circ \tau$ and $\phi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$. Similarly to some of the arguments we used in Lemma 2.2, consider expanding the k^{th} derivative of $\sigma \circ \psi \circ \tau$ at z_0 with chain rule and product rule. Then use the assumptions to simplify.

Example (3.2)**.** *Let T_n be the n^{th} Chebyshev polynomial of the first kind [12]. Let ψ be any function holomorphic at the origin with power series expansion $\psi(z) = 1 - \frac{1}{2}z^2 + \mathcal{O}(z^3)$ (using big $\mathcal{O}$ notation as $z \to 0$). Then the sequence of functions $\left\{T_n^{\circ k} \circ \psi\left(\frac{z}{n^k}\right)\right\}_{k=0}^{\infty}$ has uniform-on-compacts convergence to $\cos z$ as $k \to \infty$ for all $n \geq 2$.*

Result 1 is a special case of this example when $n = 2$. Other values of n represent other multiple angle identities for $\cos z$. The Chebyshev polynomials satisfy $T_n(\cos z) = \cos nz$ by definition. ψ is an approximation to $\cos z$ near the origin. An explicit formula for T_n is given below [12].

$$T_n(z) = n \sum_{k=0}^{n} 2^k \frac{(n+k-1)!}{(n-k)!(2k)!}(x-1)^k$$

Take any $n \geq 2$. To invoke Theorem 2.14, let $z_0 = 0$, $U = \mathbb{C}$, $\tau(z) := \frac{z}{n}$, $\sigma := T_n$, and $m = 3$. Then $w_0 := \psi(0) = 1$ and $\lambda := \tau'(0) = \frac{1}{n}$. Let us determine $\sigma'(w_0) = T_n'(1)$. This is precisely the coefficient of $(x-1)$ in the formula, so $T_n'(1) = n^2$. Then $|\lambda|^m |\sigma'(w_0)| = \frac{1}{n}$, which is less than one, as desired. Since $\psi(z) = 1 - \frac{1}{2}z^2 + \mathcal{O}(z^3)$ and $\cos z = T_n\left(\cos\left(\frac{z}{n}\right)\right)$, it follows that $\psi_0^{(k)}(0) = \psi_1^{(k)}(0)$ for all $k < m$.

The required conditions have been verified. So the sequence of functions converges to some ϕ. By uniqueness, $\phi = \cos$.

Example (3.3)**.** *Let ψ be any function holomorphic at the origin with power series expansion $\psi(z) = z + \mathcal{O}(z^2)$. Define $\sigma(z) := \frac{2z}{1-z^2}$. Then $\{\sigma^{\circ k} \circ \psi\left(\frac{z}{2^k}\right)\}_{k=0}^{\infty}$ converges to $\tan z$ uniformly on all compact sets not containing singularities of $\tan z$ (which are $\frac{\pi}{2}(2n+1)$ for $n \in \mathbb{Z}$). The sequence converges to ∞ on the singularities of $\tan z$, which means: for all singularities z and for all $r > 0$, there exists $n \in \mathbb{N}$ such that $\left|\sigma^{\circ k} \circ \psi\left(\frac{z}{2^k}\right)\right| > r$ for all $k \geq n$.*

Recall the identity $\tan 2z = \sigma(\tan z)$ and invoke Theorem 2.15 with $U = \mathbb{C}$ and $m = 2$.

Many such examples can be constructed for similar identities involving other trigonometric or hyperbolic functions. It is not always obvious how small m can be chosen when invoking our theorems. In Example 3.2, we had to determine the derivative of all Chebyshev polynomials of the first kind evaluated at $w_0 = 1$. As it turns out, such computations are unnecessary when we already have knowledge of ϕ. It is not a coincidence that $m = 3$ worked for every Chebyshev polynomial. Intuitively, we might conjecture that ψ should only need to match ϕ to the first non-zero derivative. This allows us to state a more precise version of Corollary 3.1.

Theorem (3.4)**.** *Suppose ϕ and τ are holomorphic at $z_0 \in \mathbb{C}$. Suppose σ is holomorphic at $w_0 := \phi(z_0)$. Let $\lambda := \tau'(z_0)$. Suppose $\tau(z_0) = z_0$, $\sigma(w_0) = w_0$, and $0 < |\lambda| < 1$. Additionally suppose ϕ is non-trivial (i.e. non-constant) and $\phi = \sigma \circ \phi \circ \tau$ in some neighborhood of z_0.*

Let ℓ give the first non-zero derivative of ϕ at z_0, meaning $\ell \equiv \inf\left\{\, k \in \mathbb{Z}^+ \;\middle|\; \phi^{(k)}(z_0) \neq 0 \,\right\}$, which exists by the assumption that ϕ is non-trivial. Take any ψ holomorphic at z_0 such that $\phi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k \leq \ell$ (including $k = 0$). Then there exists a neighborhood $\mathcal{D}$ of z_0 such that $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ uniformly converges to ϕ on $\mathcal{D}$ as $n \to \infty$.

We intend to invoke Theorem 2 with $m := \ell + 1$, so it suffices to show $\lambda^{\ell}\sigma'(w_0) = 1$. We take the ℓ^{th} derivative at z_0 of each side of the functional equation $\phi = \sigma \circ \phi \circ \tau$ using chain rule and product rule. We inspect the derivatives to determine a pattern.

$$\phi'(z) = \sigma'\Big(\phi\big(\tau(z)\big)\Big)\phi'\big(\tau(z)\big)\tau'(z)$$

$$\phi''(z) = \sigma''\Big(\phi\big(\tau(z)\big)\Big)\phi'\big(\tau(z)\big)^2\tau'(z)^2 + \sigma'\Big(\phi\big(\tau(z)\big)\Big)\phi''\big(\tau(z)\big)\tau'(z)^2 + \sigma'\Big(\phi\big(\tau(z)\big)\Big)\phi'\big(\tau(z)\big)\tau''(z)$$

When we plug in $z = z_0$, every term with a factor $\phi^{(k)}\big(\tau(z)\big)$ for $k < \ell$ will go to zero. When we take the ℓ^{th} derivative and fully expand with product/chain rule, only a single term on the right side of the equation will have the factor $\phi^{(\ell)}\big(\tau(z)\big)$, and all other terms will go to zero. At each step in the process of taking derivatives, we can effectively ignore all terms but the one with the highest derivative of ϕ. Writing a few steps out, we have:

$$\phi'(z_0) = \sigma'\Big(\phi\big(\tau(z_0)\big)\Big)\phi'\big(\tau(z_0)\big)\tau'(z_0)$$

$$\phi''(z_0) = \sigma'\Big(\phi\big(\tau(z_0)\big)\Big)\phi''\big(\tau(z_0)\big)\tau'(z_0)^2$$

$$\vdots$$

$$\phi^{(\ell)}(z_0) = \sigma'\Big(\phi\big(\tau(z_0)\big)\Big)\phi^{(\ell)}\big(\tau(z_0)\big)\tau'(z_0)^{\ell}$$

Dividing by $\phi^{(\ell)}(z_0)$ in the last equation proves $\lambda^{\ell}\sigma'(w_0) = 1$ as desired. Hence we are done.

Our next goal is to construct a composition expansion for $\cos^{-1} x$ going from $[-1, 1]$ to $[0, \pi]$. Since $[-1, 1]$ is not an open subset of $\mathbb{C}$ (required for our previous extension theorems), we state and prove another convergence extension.

Theorem (3.5)**.** *Suppose ϕ and τ are holomorphic at $z_0 \in \mathbb{C}$. Suppose σ is holomorphic at $w_0 := \phi(z_0)$. Let $\lambda := \tau'(z_0)$. Suppose $\tau(z_0) = z_0$, $\sigma(w_0) = w_0$, and $0 < |\lambda| < 1$. Take any set $U \subset \mathbb{C}$ such that $\tau(U) \subset U$ and $\lim_{n\to\infty} \tau^{\circ n}(z) = z_0$ for all $z \in U$. Suppose $\phi = \sigma \circ \phi \circ \tau$ on the union of U with a neighborhood of z_0. Suppose σ is continuous on a set $V \subset \mathbb{C}^*$ such that $\phi(U) \subset V$ and $\sigma(V) \subset V$.*

Take the smallest $m \in \mathbb{Z}^+$ such that $|\lambda|^m |\sigma'(w_0)| < 1$. Equivalently, if ϕ is non-trivial, $m \equiv 1+\inf \{ k \in \mathbb{Z}^+ \mid \phi^{(k)}(z_0) \neq 0 \}$. Take any ψ holomorphic at z_0 such that $\phi^{(k)}(z_0) = \psi^{(k)}(z_0)$ for all $k < m$ (including $k = 0$). Then $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ has pointwise convergence to ϕ on U as $n \to \infty$.

(Remark) The domains and codomains of the functions in question are $\tau : U \to U$, $\phi : U \to V$, and $\sigma : V \to V$. Technically, we also need τ, ϕ defined in neighborhoods of z_0, and σ defined in a neighborhood of w_0, but those neighborhoods are irrelevant for the point we are about to make. Recall $V \subset \mathbb{C}^* := \mathbb{C} \cup \{\infty\}$. We briefly describe/define what it means for σ to be continuous when $\infty \in V$. If $\sigma(\infty) \neq \infty$, then continuity of σ at ∞ means: for all $\epsilon > 0$, there exists $r > 0$ such that $|\sigma(z) - \sigma(\infty)| < \epsilon$ for all $z \in V$ with $|z| > r$. If $\sigma(w) = \infty$ for some $w \neq \infty$, then continuity of σ at w means: for all $s > 0$, there exists $\delta > 0$ such that $|\sigma(z)| > s$ for all $z \in V$ with $|w - z| < \delta$. If $\sigma(\infty) = \infty$, then continuous of σ at ∞ means: for all $s > 0$, there exists $r > 0$ such that $|\sigma(z)| > s$ for all $z \in V$ with $|z| > r$.

Invoke Theorem 2 to get a neighborhood $\mathcal{D}$ of z_0 such that $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ uniformly converges to ϕ on $\mathcal{D}$. Take any $z \in U$. By the assumption $\lim_{n\to\infty} \tau^{\circ n}(z) = z_0$, there exists $j \in \mathbb{N}$ such that $\tau^{\circ j}(z) \in \mathcal{D}$. Since $\tau(U) \subset U$, we also have $\tau^{\circ j}(z) \in U$. Since σ is continuous on $V \supset \phi(U)$ and $\sigma(V) \subset V$, it must be that $\sigma^{\circ j}$ is continuous at $\phi \circ \tau^{\circ j}(z)$. Combine these facts to prove the desired pointwise convergence.

$$\phi(z) = \sigma^{\circ j} \circ \phi \circ \tau^{\circ j}(z) = \sigma^{\circ j} \left(\lim_{n\to\infty} \sigma^{\circ n} \circ \psi \circ \tau^{\circ(n+j)}(z) \right) = \lim_{n\to\infty} \sigma^{\circ(n+j)} \circ \psi \circ \tau^{\circ(n+j)}(z) = \lim_{n\to\infty} \sigma^{\circ n} \circ \psi \circ \tau^{\circ n}(z)$$

If $\phi(w) = \infty$ for some $w \in U$, then the sequence $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}(w)$ converges to ∞ as described in Example 3.3. Hence the theorem is proven.

The composition expansion for $\cos^{-1} x$ is decidedly less useful due to all the nested square roots, but we present the argument below merely to show it is possible.

Example (3.6)**.** *Let ψ be any function analytic at $x = 1$ with power series expansion $\psi(x) = -2(x-1) + \mathcal{O}\left((x-1)^2\right)$. For any $x \in [-1, 1]$, define $\tau(x) := \sqrt{\frac{1+x}{2}}$. Then $\lim_{k\to\infty} 2^k \sqrt{\psi \circ \tau^{\circ k}(x)} = \cos^{-1}(x)$ for all $x \in [-1, 1]$ where $y = \cos^{-1}(x)$ is the unique value in $[0, \pi]$ such that $\cos y = x$.*

Considering the power series expansion of $\cos z$ about the origin, there exists an entire function f such that $f(z^2) = \cos z$ for all $z \in \mathbb{C}$. In particular, $f(z) = 1 - \frac{z}{2!} + \frac{z^2}{4!} - \frac{z^3}{6!} + \cdots$. From the identity $\cos 2z = 2\cos^2 z - 1$, we get $f(4z^2) = 2f^2(z^2) - 1$ for all $z \in \mathbb{C}$. Since $z \mapsto z^2$ is a surjective map, we have $f(4z) = 2f^2(z) - 1$ for all $z \in \mathbb{C}$.

Since $f(0) = 1$ and $f'(0) = -\frac{1}{2}$, we can locally invert f about the origin. So we have a holomorphic inverse $f^{-1}(z) = -2(z-1) + \mathcal{O}\left((z-1)^2\right)$ defined in some neighborhood of $z = 1$. Notice ψ agrees with f^{-1} to the first non-zero derivative at $z = 1$. Then $4f^{-1}(z) = f^{-1}(2z^2 - 1)$ in a neighborhood of $z = 1$. Let $g(z) = 2z^2 - 1$. Since $g'(1) = 4$, we can locally invert g about $z = 1$, which means $4f^{-1} \circ g^{-1}(z) = f^{-1}(z)$ in a neighborhood of $z = 1$. Note the derivative of $g^{-1}(z)$ at $z = 1$ is $\lambda = \frac{1}{4}$. On the interval $[-1, 1]$, τ agrees with g^{-1}, so let us write τ and g^{-1} interchangably.

We could invoke Theorem 3.4 to prove $4^k \psi \circ \tau^{\circ k}$ converges to f^{-1} uniformly in some neighborhood of $z = 1$. Considering the relation $f(x^2) = \cos x$ with the fact that $\cos x$ is invertible on $[0, \pi]$, it must be that f is invertible on $[0, \pi^2]$. So we can extend our local inverse f^{-1} to a function going from $[-1, 1]$ to $[0, \pi^2]$. We must have $f^{-1}(x) = \left(\cos^{-1} x\right)^2$ for all $x \in [-1, 1]$.

Next we use Theorem 3.5 to extend convergence of $4^k \psi \circ \tau^{\circ k}(x)$ to $\left(\cos^{-1} x\right)^2$ on all of $U := [-1, 1]$. In this case, $\sigma(z) = 4z$, so continuity is immediately satisfied. To complete the proof, we need to show $\lim_{n\to\infty} \tau^{\circ n}(x) = 1$ for all $x \in [-1, 1]$. This is relatively easy to verify, so I'll just outline the proof argument. The only fixed point is $\tau(1) = 1$, and the sequence $\{\tau^{\circ n}(x)\}_{n=0}^{\infty}$ is monotonically increasing for every $x \in [-1, 1]$, so the desired statement immediately follows.

Thus we've proven $\lim_{k\to\infty} 4^k \psi \circ \tau^{\circ k}(x) = \left(\cos^{-1}(x)\right)^2$ for all $x \in [-1, 1]$. Taking the square root of both sides and moving the limit proves the desired result: $\lim_{k\to\infty} 2^k \sqrt{\psi \circ \tau^{\circ k}(x)} = \cos^{-1}(x)$ for all $x \in [-1, 1]$.

We had to use the trick with $f(z^2) = \cos z$ because $\cos z$ is not locally invertible about the origin (while f is). Comparing this example to our previous work, notice that $\lim_{k\to\infty} 2^k \sqrt{\psi \circ \tau^{\circ k}(\cos x)} = x$ with $\psi(x) = 2 - 2x$ is the result of formally inverting Result 1 (i.e. ignore the limit and imagine trying to solve for $\cos x$). I pulled the next example from [5].

Example (3.7)**.** *Let ψ be any function holomorphic at the origin which satisfies $\psi(0) = 1$. Define $\sigma(z) := 4z^2 - 3z$ and $\tau(z) := \frac{z}{5}$. Then $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ has uniform-on-compacts convergence as $n \to \infty$ to an entire function ϕ which satisfies $\phi(0) = 1$, $\phi'(0) = \psi'(0)$, and $\phi(5z) = 4\phi^2(z) - 3\phi(z)$.*

We intend to invoke Theorem 2.14. Let $z_0 = 0$ and $w_0 = 1$. Notice that $\tau(z_0) = z_0$ and $\sigma(w_0) = w_0$. We have $\lambda = \tau'(z_0) = \frac{1}{5}$ and $\sigma'(w_0) = 5$, so $m = 2$ works. Let $\psi_1 = \sigma \circ \psi \circ \tau$. The last condition to verify is that $\psi'(0) = \psi_1'(0)$.

$$\psi_1'(0) = \sigma'\Big(\psi\big(\tau(0)\big)\Big)\psi'\big(\tau(0)\big)\tau'(0) = \sigma'(1)\psi'(0)\tau'(0) = 5\psi'(0)\left(\frac{1}{5}\right) = \psi'(0)$$

Thus we are done. The interesting thing about this example is that we didn't have to specify the first derivative $\psi'(0)$ beforehand. In general, this suggests we never have to worry about matching the highest derivative (i.e. we can effectively remove assumptions from the statement of Theorem 2).

Theorem (3.8)**.** *Suppose ψ and τ are holomorphic at $z_0 \in \mathbb{C}$. Suppose σ is holomorphic at $w_0 := \psi(z_0)$. Let $\lambda := \tau'(z_0)$. Suppose $\tau(z_0) = z_0$, $\sigma(w_0) = w_0$, and $0 < |\lambda| < 1$. Suppose there exists $\ell \in \mathbb{Z}^+$ such that $\sigma'(w_0) = \lambda^{-\ell}$ and $\psi^{(k)}(z_0) = 0$ for all $k \in \mathbb{N}$ with $0 < k < \ell$.*

Then there exists a neighborhood $\mathcal{D}$ of z_0 such that $\tau(\mathcal{D}) \subset \mathcal{D}$ where $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ converges uniformly on $\mathcal{D}$ to a holomorphic function ϕ as $n \to \infty$. Furthermore, ϕ is the unique holomorphic function on $\mathcal{D}$ satisfying $\phi(z_0) = w_0$, $\phi^{(\ell)}(z_0) = \psi^{(\ell)}(z_0)$, and $\phi = \sigma \circ \phi \circ \tau$. It is also true that $\phi^{(k)}(z_0) = 0$ for all $k \in \mathbb{N}$ with $0 < k < \ell$, but it unnecessary to check these intermediary derivatives for uniqueness.

By Theorem 3.4, we haven't lost any generality by converting to these assumptions (i.e. any non-trivial ϕ admitting a composition expansion is guaranteed to satisfy the assumptions stated in this theorem). It is much easier to verify these conditions, since we no longer need to computationally match any derivatives.

For the proof, invoke Theorem 2 with $m := \ell + 1$. To prove the first ℓ derivatives of ψ and $\sigma \circ \psi \circ \tau$ match at z_0, use the chain rule and product rule similarly to what we did with the proof of Theorem 3.4. For $k < \ell$, it should be relatively obvious that the k^{th} derivative of $\sigma \circ \psi \circ \tau$ is zero at z_0. For matching the ℓ^{th} derivative, use the assumption $\lambda^\ell \sigma'(w_0) = 1$.

Now we explain why we don't have to check the intermediary derivatives for uniqueness of ϕ. Assume ϕ is a holomorphic function on $\mathcal{D}$ which satisfies $\phi(z_0) = w_0$, $\phi^{(\ell)}(z_0) = \psi^{(\ell)}(z_0)$, and $\phi = \sigma \circ \phi \circ \tau$. We prove $\phi^{(k)}(z_0) = 0$ for all $0 < k < \ell$. Assume for contradiction that $\phi^{(k)}(z_0) \neq 0$ for some $0 < k < \ell$. Then we have $j := \inf\left\{\, k \in \mathbb{Z}^+ \;\middle|\; \phi^{(k)}(z_0) \neq 0 \,\right\}$ where $j \neq \ell$. Using an argument from the proof of Theorem 3.4, we can prove $\sigma'(w_0) = \lambda^{-j}$, contradicting that $\sigma'(w_0) = \lambda^{-\ell}$.

If we want to find non-trivial ϕ such that $\phi = \sigma \circ \phi \circ \tau$ for given σ, τ, then this theorem may be a good place to start. Find all pairs (z_0, w_0) such that $\tau(z_0) = z_0$, $\sigma(w_0) = w_0$, $0 < |\tau'(z_0)| < 1$, and $\sigma'(w_0)\tau'(z_0)^\ell = 1$ for some $\ell \in \mathbb{Z}^+$. For any such pair, there must exist non-trivial ϕ in a neighborhood of z_0 satisfying $\phi(z_0) = w_0$ and $\phi = \sigma \circ \phi \circ \tau$. In particular, $\phi^{(\ell)}(z_0)$ can be arbitrarily set to any desired value.

Additionally, if $\tau(z) = \lambda(z - z_0)$, then $\phi^{(k)}(z_0) = 0$ whenever k is not a multiple of ℓ. This can be seen from looking at the power series coefficient recurrence relation induced by $\phi = \sigma \circ \phi \circ \tau$ (c.f. Lemma 2.7 and [5, Lemma 2.1]). This implies there exists a function f holomorphic at the origin such that $\phi(z) = f\left((z - z_0)^\ell\right)$ for all z sufficiently close to z_0.

We restate the convergence extensions Theorem 2.14 and Theorem 2.15 with refined assumptions; however, the old versions are not necessarily obselete. The refined assumptions are equivalent when ϕ is non-trivial, but there are cases which require the old assumptions when ϕ is trivial (e.g. $\sigma'(w_0) \neq \lambda^{-\ell}$ for all $\ell \in \mathbb{Z}^+$).

Theorem (3.9)**.** *Suppose ψ is holomorphic at $z_0 \in \mathbb{C}$. Suppose τ is holomorphic on an open set $U \subset \mathbb{C}$ containing z_0. Suppose σ is an entire function. Let $w_0 := \psi(z_0)$ and $\lambda := \tau'(z_0)$. Suppose $\tau(U) \subset U$, $\tau(z_0) = z_0$, $\sigma(w_0) = w_0$, and $0 < |\lambda| < 1$. Suppose $\lim_{n\to\infty} \tau^{\circ n}(z) = z_0$ uniformly on compacts $K \subset U$. Suppose there exists $\ell \in \mathbb{Z}^+$ such that $\sigma'(w_0) = \lambda^{-\ell}$ and $\psi^{(k)}(z_0) = 0$ for all $0 < k < \ell$.*

Then $\{\psi_n\}_{n=0}^{\infty}$ converges uniformly on compacts $K \subset U$ to a holomorphic function ϕ. Furthermore, ϕ is the unique function on U which is holomorphic at z_0 and satisfies $\phi(z_0) = w_0$, $\phi^{(\ell)}(z_0) = \psi^{(\ell)}(z_0)$, and $\phi = \sigma \circ \phi \circ \tau$. It is also true that $\phi^{(k)}(z_0) = 0$ for all $0 < k < \ell$.

Theorem (3.10)**.** *Suppose ψ is holomorphic at $z_0 \in \mathbb{C}$. Suppose τ is holomorphic on an open set $U \subset \mathbb{C}$ containing z_0. Suppose σ is a rational function (quotient of polynomials). Let $w_0 := \psi(z_0)$ and $\lambda := \tau'(z_0)$. Suppose $\tau(U) \subset U$, $\tau(z_0) = z_0$, $\sigma(w_0) = w_0$, and $0 < |\lambda| < 1$. Suppose $\lim_{n\to\infty} \tau^{\circ n}(z) = z_0$ uniformly on compacts $K \subset U$. Suppose there exists $\ell \in \mathbb{Z}^+$ such that $\sigma'(w_0) = \lambda^{-\ell}$ and $\psi^{(k)}(z_0) = 0$ for all $0 < k < \ell$.*

Then $\{\psi_n\}_{n=0}^{\infty}$ has pointwise convergence to a meromorphic function ϕ on U and the limit occurs uniformly on all compacts $K \subset U$ not containing singularities of ϕ. Furthermore, ϕ is the unique function $U \to \mathbb{C}^*$ which is holomorphic at z_0 and satisfies $\phi(z_0) = w_0$, $\phi^{(\ell)}(z_0) = \psi^{(\ell)}(z_0)$, and $\phi = \sigma \circ \phi \circ \tau$. It is also true that $\phi^{(k)}(z_0) = 0$ for all $0 < k < \ell$.

Conclusion

This paper has sought to address convergence of $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ to another function ϕ as $n \to \infty$ when τ is analytic at an attracting fixed point z_0, and σ is analytic at a repelling fixed point $w_0 := \psi(z_0)$. ψ may be considered an approximation of ϕ near z_0, and ϕ satisfies $\phi = \sigma \circ \phi \circ \tau$. The sequence $\sigma^{\circ n} \circ \psi \circ \tau^{\circ n}$ may be referred to as a composition expansion of ϕ about z_0. As expected, composition expansions are relatively well-behaved for analytic functions. It is possible to extend these ideas to more general settings (e.g. metric spaces with a notion of asymptotic equivalence at z_0 between ϕ and ψ), but this complicates the behavior of expansions.

Most of the heavy lifting went into the proof of Theorem 2. Everything else followed from fleshing out consequences. Theorem 3.4 and Theorem 3.8 appear to be the most elegant of the results presented here. When an expression for ϕ is already known, Theorem 3.4 is ideal for constructing composition expansions of ϕ. When an expression for ϕ is *a priori* not known, Theorem 3.8 is ideal for constructing ϕ. Theorem 3.5, Theorem 3.9, and Theorem 3.10 are great for extending convergence of composition expansions; however, Theorem 3.5 is typically useful only when an expression for ϕ is *a priori* known.

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